Description of the Orbit with Central Forces

It is straightforward to transform the equations of motion for $r(t)$ and $\theta(t)$ to an equation for the trajectory $r(\theta)$.

$$\frac{d\theta}{dt} = \frac{\ell}{\mu r^2} \Rightarrow \frac{d}{dt} f(r,\theta) = \frac{\ell}{\mu r^2} \frac{d}{d\theta} f(r,\theta)$$

In a central force $f(r) = -\frac{\partial V}{\partial r}$, we found that $r(t)$ satisfies the equation of motion

$$\mu \ddot{r} - \frac{\ell^2}{\mu r^3} = f(r)$$

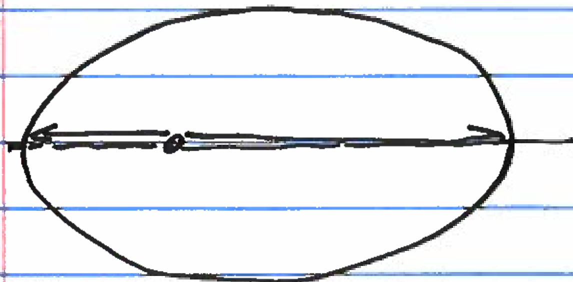
$$\Rightarrow \frac{\ell}{r^2} \frac{d}{d\theta}\left(\frac{\ell}{\mu r^2} \frac{dr}{d\theta} \right) - \frac{\ell^2}{\mu r^3} = f(r)$$

Define $u = 1/r$.

Exercise $\rightarrow$ $$\boxed{\frac{d^2 u}{d\theta^2} + u = -\frac{\mu}{\ell^2} \frac{d}{du} V\left(\frac{1}{u}\right)}$$

Suppose $\theta = 0$ is a turning pt. for $r(\theta)$. At the turning pt. $u = u(0)$, $\left.\frac{du}{d\theta}\right|_{\theta=0} = 0$.

The equation of motion and initial conditions are invariant under $\theta \to -\theta$. Hence, the orbit is invariant under reflection about the apsidal vectors, the displacements of the turning pts. from the origin $r=0$.



Apsides = pts at which $r(\theta)$ is maximum or minimum, $\left.\frac{dr}{d\theta}\right|_{r_{aps.}} = 0$

From conservation of energy,

$$\frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + V(r) = E$$

$$\rightarrow \quad \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - V(r) - \frac{\ell^2}{2\mu r^2}\right)}$$

$$= \frac{\ell}{\mu r^2}\frac{dr}{d\theta}$$

$$\Rightarrow \theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu V(r)}{\ell^2} - \frac{1}{r^2}}} + \theta_0$$

$$\xrightarrow{u = 1/r} \boxed{\theta = -\int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu V(\frac{1}{u})}{\ell^2} - u^2}} + \theta_0}$$

This determines $\theta(u)$, which can be inverted to find $u(\theta)$, and then $r(\theta) = 1/u(\theta)$.

For power law potentials $V(r) = +\frac{K}{n+1} r^{n+1}$

$\theta(u)$ is expressible in terms of trig functions for $n = 1, -2, -3$.

$\theta(u)$ is expressible in terms of elliptic functions for $n = 5, 3, 0, -4, -5, -7$

Closed Orbits and Bertrand's Theorem

Consider circular orbits first:

with $V_{eff}(r) = V(r) + \frac{\ell^2}{2\mu r^2}$, there is a circular orbit for r_0 such that $\left.\frac{dV_{eff}}{dr}\right|_{r=r_0} = 0$, or

$$f(r_0) \equiv -\frac{dV}{dr_0} = -\frac{\ell^2}{\mu r_0^3} < 0 \leftarrow \text{attractive force.}$$

→ For any attractive force, $\exists$ circular orbit for any r_0, with corresponding ℓ.

Stable orbit: $\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0} = -\left.\frac{df}{dr}\right|_{r=r_0} + \frac{3\ell^2}{\mu r_0^4} > 0$

$$\rightarrow \left.\frac{df}{dr}\right|_{r=r_0} < -\frac{3f(r_0)}{r_0}$$

$$\left.\frac{d\ln f}{d\ln r}\right|_{r=r_0} > -3$$

Power law force $f(r) = -Kr^n$

Stability $\rightarrow -Knr^{n-1} < 3Kr^{n-1}$

$$\Rightarrow \boxed{n > -3} \text{ for stable circular orbits } \forall r.$$

Perturb about stable circular orbit:

$$\frac{d^2 u}{d\theta^2} = -\frac{\mu}{\ell^2}\frac{1}{u^2} f\left(\frac{1}{u}\right) - u$$

Circular orbit: $f\left(\frac{1}{u_0}\right) = -\frac{\ell^2}{\mu} u_0^3$, or

$$f(r_0) = -\frac{\ell^2}{\mu r_0^3}$$

Let $u = u_0 + \Delta u(\theta)$

$$\frac{d^2 u}{d\theta^2} = \frac{d^2(\Delta u)}{d\theta^2} = -\frac{\mu}{\ell^2}\frac{1}{(u_0+\Delta u)^2} f\left(\frac{1}{u_0+\Delta u}\right) - (u_0+\Delta u)$$

$$= \cancel{\left[-\frac{\mu}{\ell^2}\frac{1}{u_0^2} f\left(\frac{1}{u_0}\right) - u_0\right]}$$

$$+ \Delta u\left(-1 + \underbrace{\frac{\mu}{\ell^2}\frac{2}{u_0^3} f\left(\frac{1}{u_0}\right)}_{-2} + \frac{\mu}{\ell^2 u_0^4} f'\left(\frac{1}{u_0}\right)\right)$$

$$+ O(\Delta u)^2$$

$$\boxed{\frac{d^2 \Delta u}{d\theta^2} \approx \Delta u\left(-3 + \frac{\mu}{\ell^2 u_0^4} f'\left(\frac{1}{u}\right)\Big|_{u_0}\right)}$$

This equation describes harmonic motion, with solution of the form

$$u = u_0 + a\cos\beta\theta$$

for some a, β.

Suppose a power law force $f(r) = -Kr^n$

$$\frac{df}{dr} = -Knr^{n-1} = \frac{n}{r} f(r)$$

$$f(r_0) = -\frac{\ell^2}{\mu r_0^3} \rightarrow \frac{-\mu}{\ell^2 u_0^4} = \frac{r_0}{f(r_0)}$$ (Recall $u = 1/r$)

$$\frac{d^2 \Delta u}{d\theta^2} = \Delta u \left(-3 - \frac{r_0}{f(r_0)} f'(r_0) \right)$$

with $\Delta u = a \cos \beta\theta$, we have

$$\boxed{\beta^2 = 3 + \frac{r_0}{f(r_0)} f'(r_0)}$$

If $\beta = \frac{p}{q}$ is a rational #, then after q revolutions the orbit closes.

Question: For what force laws is β the same rational number for all circular orbits?

$$\beta^2 = 3 + \frac{d \ln f}{d \ln r} \Rightarrow \boxed{f(r) = -\frac{K}{r^{3-\beta^2}}}$$

Any rational $\beta \rightarrow$ force law with closed stable orbits close to circular orbit.

Bertrand's Theorem: For larger deviations from circular orbits, the only forces w/ closed orbits of arbitrary bound particles are $\beta^2 = 1$ $\left(f(r) = -\frac{K}{r^2} \right)$ and $\beta^2 = 4$ $\left(f(r) = -Kr \right)$

Inverse-Square Force: Complete Solution

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu K u}{\ell^2} - u^2}}$$

constant - determined by initial conditions

The integral can be done analytically:

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}\left(- \frac{\beta + 2\gamma x}{\sqrt{q}} \right)$$

where $q = \beta^2 - 4\alpha\gamma$.

with $\alpha = \frac{2\mu E}{\ell^2}$, $\beta = \frac{2\mu K}{\ell^2}$, $\gamma = -1$

$$\Rightarrow q = \left(\frac{2\mu K}{\ell^2}\right)^2 \left(1 + \frac{2E\ell^2}{\mu K^2}\right)$$

$$\theta = \theta' - \cos^{-1}\left(\frac{\frac{\ell^2 u}{\mu K} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu K^2}}} \right)$$

Solve for $u = 1/r$:

$$\boxed{\frac{1}{r} = \frac{\mu K}{\ell^2}\left(1 + \sqrt{1 + \frac{2E\ell^2}{\mu K^2}} \cos(\theta - \theta')\right)}$$

Conic section with focus at the origin:

$$\frac{1}{r} = C\left[1 + e\cos(\theta - \theta')\right]$$

we identify $e = \sqrt{1 + \frac{2E\ell^2}{\mu K^2}}$ for our solutions.

$e > 1$, $E > 0$ hyperbola
$e = 1$, $E = 0$ parabola
$e < 1$, $E < 0$ ellipse
$e = 0$, $E = -\frac{\mu K^2}{2\ell^2}$ circle.

Major axis of elliptical orbits depends on energy:

$$E - \frac{\ell^2}{2\mu r^2} + \frac{K}{r} = 0$$ ← determines apsidal distances.

Two solutions: r_1, r_2

write the equation for r_1, r_2 as $r^2 + \frac{K}{E} r - \frac{\ell^2}{2\mu E}$

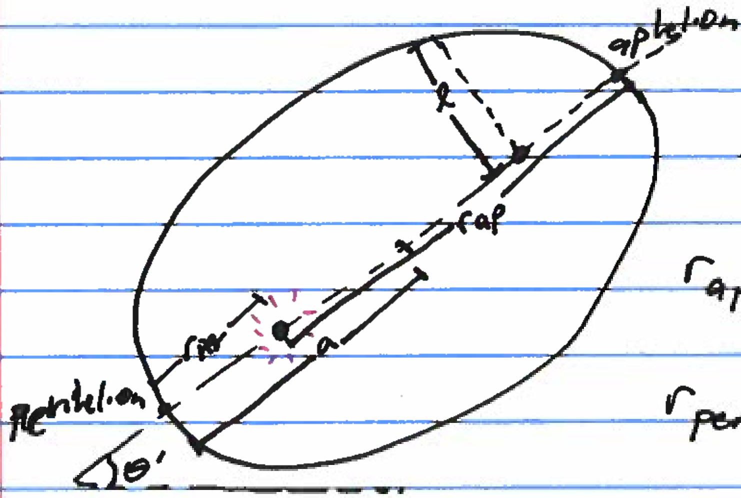
$$r = -\frac{K}{2E} \pm \frac{1}{2}\sqrt{\left(\frac{K}{E}\right)^2 + \frac{2\ell^2}{\mu E}}$$

$$a = \frac{r_1 + r_2}{2} = -\frac{K}{2E}$$ semimajor axis.

Eccentricity $e = \sqrt{1 - \frac{\ell^2}{\mu K a}}$

we have $\frac{\ell^2}{\mu K} = a(1-e^2)$

$$\rightarrow \boxed{r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta')}}$$



$\ell = a(1-e^2)$ semilatus rectum

$r_{ap} = (1+e)a$ farthest point from focus

$r_{per} = (1-e)a$ closest point to focus

Eccentricity $e = \frac{r_{ap} - r_{per}}{r_{ap} + r_{per}}$

Semimajor axis $a = \frac{r_{ap} + r_{per}}{2}$

Time to complete orbit

$$\frac{d\theta}{dt} = \frac{\ell}{\mu r^2}. \quad \text{Substituting for } r(\theta),$$

$$\Rightarrow \tau = \frac{\ell^3}{\mu K^2} \int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{(1+e\cos(\theta-\theta_0))^2}$$

period of orbit

$$\overset{\text{Exercise}}{=} 2\pi \sqrt{\frac{\mu}{K}}\, a^{3/2}$$

$$\boxed{\tau^2 \propto a^3 \quad \text{Kepler's 3rd Law}}$$

with $\mu = \dfrac{m_1 m_2}{m_1+m_2}$, $K = G m_1 m_2$

$$\boxed{\tau = \frac{2\pi a^{3/2}}{\sqrt{G(m_1+m_2)}}} \approx \frac{2\pi a^{3/2}}{\sqrt{G m_2}}$$

for planets much less massive than the sun.

mass of planet

mass of sun

The appearance of m_1+m_2 violates the universality of Kepler's 3rd law. For planets light compared to the sun, the constant of proportionality is approximately the same for those planets.