Variational Principles and Lagrange's Eqs.
Goldstein - Ch. 2

We arrived at Lagrange's Eqs. via D'Alembert's principle by considering small virtual displacements from the instantaneous state of the system.

Instead we can consider small changes to the entire path of the system, in configuration space $\{q_1(t), q_2(t) \dots, q_n(t)\}$.

Hamilton's principle applies to systems for which all applied forces are derivable from a generalized potential $V(q_i, \dot{q}_i, t)$.

The generalized potential can depend on velocities.

Such systems are called monogenic.

☆ Hamilton's Principle: The motion of a system from time t_1 to t_2 is such that the action $I = \int_{t_1}^{t_2} L\, dt$, where $L = T - V$, has a stationary value.

Lagrangian

i.e. The motion is such that $\delta I = \delta \int_{t_1}^{t_2} L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t)\,dt = 0.$

If the system constraints are holonomic, then Hamilton's principle $\leftrightarrow$ Lagrange's Eqs.

Calculus of Variations

Determination of the path that stationarizes the action relies on the calculus of variations

The general problem:

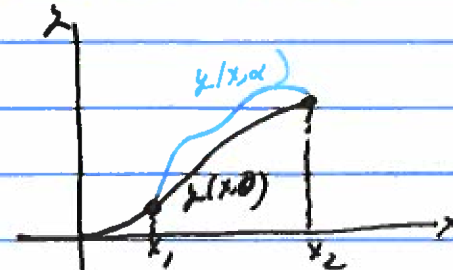
Given a path $y(x)$, suppose we have a function $f(y, \dot{y}, x)$, where $\dot{y} = \frac{dy}{dx}$.

We want to study how the integral $J \equiv \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx$ varies with the path $y(x)$.

We only consider paths with $y(x_1)$ and $y(x_2)$ fixed. (For now.)

Consider some function $\eta(x)$ — continuous, nonsingular in the region $x_1 \le x \le x_2$.

Define $y(x, \alpha) \equiv y(x, 0) + \alpha\, \eta(x)$.



As $\alpha \to 0$, $y(x,\alpha) \to y(x)$.

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)\, dx.$$

Stationary point: $\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0 \rightarrow y(x,0)$ stationarizes $J(\alpha)$ over variations in path of this type.

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} \right) dx$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x\,\partial \alpha} \right) dx$$

Integrate by parts

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha} \right) dx + \frac{\partial f}{\partial \dot{y}}\cancel{\frac{\partial y}{\partial \alpha}}\Big|_{x_1}^{x_2}$$

$\frac{\partial y}{\partial \alpha} = 0$ at $x = x_1$ and x_2.

$$\Rightarrow \frac{dJ}{d\alpha}\Big|_{\alpha=0} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right) \frac{\partial y}{\partial \alpha}\Big|_{\alpha=0} dx = 0 \quad \text{stationarity}$$

Define $\delta y \equiv \frac{\partial y}{\partial \alpha}\Big|_{\alpha=0} d\alpha = \eta(x)\, d\alpha$

$$\delta J \equiv \frac{dJ}{d\alpha}\Big|_{\alpha=0} d\alpha$$

$$\Rightarrow \delta J = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right) \delta y\, dx = 0.$$

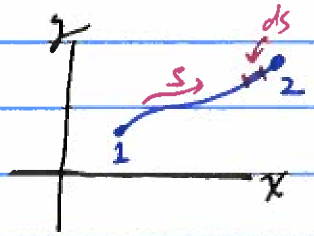
☆ Fundamental lemma of the calculus of variations:

If $\int_{x_1}^{x_2} M(x)\,\eta(x)\,dx = 0$ $\forall\, \eta(x)$ continuous (through the second derivative),

then $M(x) = 0$ bet. x_1 and x_2.

$$\Rightarrow \delta J = 0 \;\; \forall\, \delta y(x) \quad \longleftrightarrow \quad \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0}$$

Example: Shortest distance between two points on a plane:



Length element $ds = \sqrt{dx^2 + dy^2}$

Length of curve $\equiv I = \int_{pt.1}^{pt.2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Shortest path → I is stationary.

Define $f(y, \dot{y}, x) = \sqrt{1 + \dot{y}^2}$, $I = \int_{x_1}^{x_2} f\, dx$

$\dot{y} = \frac{dy}{dx}$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right)$$

$$\delta I = 0 \longrightarrow \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}\right) = 0$$

$$\longrightarrow \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = \text{const.}$$

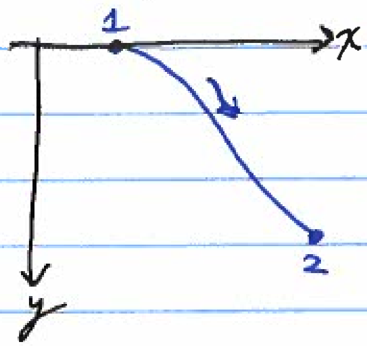
$$\longrightarrow \dot{y} = \text{const.} \longrightarrow \boxed{y = ax + b}$$ for const. a, b.

The constants a and b are determined by $y(x_1) \equiv y_1$, $y(x_2) \equiv y_2$ fixed.

⇒ Shortest path bet. two points in a plane is a straight line.

Example: The brachistochrone problem

Question: what is the curve joining two points such that a particle falling by gravity, constrained to move along the curve, travels from the higher pt. to the lower pt. in the least time?



Falling from rest: $\frac{1}{2}mv^2 = mgy$

$\rightarrow v = \sqrt{2gy}$

Time of fall: $t_{12} = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\,dx$

Define $f(y,\dot{y},x) = \sqrt{\frac{1+\dot{y}^2}{2gy}}$

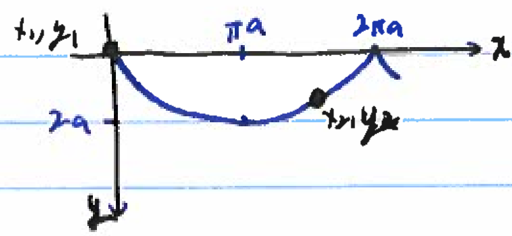
$t_{12} = \int_{x_1}^{x_2} f\,dx$ minimum with $y(x_1) = y_1$, $y(x_2) = y_2$ fixed.

$$\frac{\partial f}{\partial y} = -\frac{1}{2}\sqrt{\frac{1+\dot{y}^2}{2gy^3}}, \quad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{(1+\dot{y}^2)\cdot 2gy}}$$

$$\delta t_{12} = 0 \longrightarrow \boxed{-\frac{1}{2}\sqrt{\frac{1+\dot{y}^2}{2gy^3}} - \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{(1+\dot{y}^2)\cdot 2gy}}\right) = 0}$$

Parametric solution: $x = a(\phi - \sin\phi)$, $y = a(1-\cos\phi)$

parameter – fixed by $y(x_2) = y_2$



cycloid – same as path traced by pt. on a rolling disk

Back to Hamilton's Principle

$$I = \int_1^2 L(q_1(t), \dots, q_n(t); \dot q_1(t), \dots, \dot q_n(t); t)\, dt$$

For arbitrary smooth functions $\eta_1(t) \dots, \eta_n(t)$ between t_1 and t_2,

write
$$q_1(t,\alpha) = q_1(t,0) + \alpha\, \eta_1(t)$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$q_n(t,\alpha) = q_n(t,0) + \alpha\, \eta_n(t)$$

$$0 = \delta I = \left.\frac{\partial I}{\partial \alpha}\right|_{\alpha=0} d\alpha = \int_1^2 \sum_i \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot q_i}\frac{\partial \dot q_i}{\partial \alpha}\right)\Bigg|_{\alpha=0} d\alpha\, dx$$

Integrate by parts.

$$= \int_1^2 \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right) \underbrace{\left.\frac{\partial q_i}{\partial \alpha}\right|_{\alpha=0} d\alpha}_{\delta q_i(t)}\, dx$$

$$\delta I = \int_1^2 \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right) \delta q_i(t)\, dx = 0$$

True for all $\delta q_i(t)$

$$\rightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = 0}$$ Euler-Lagrange Eqs.

= Lagrange Eqs. if $L = T - V$.

Hamilton's Principle with Holonomic Constraints

Suppose the $q_i(t)$ are not independent, but are constrained by eqs. of the form $f_\alpha(\{q_i\}, t) = 0$, $\alpha \in \{1, \ldots, m\}$ if there are m constraints.

Then we can't vary the $q_i(t)$ independently, and the Euler-Lagrange Eqs. don't follow from Hamilton's principle.

Trick: Introduce Lagrange Multipliers λ_α, $\alpha \in \{1, \ldots, m\}$

Write $I = \int_1^2 \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt$

We treat the λ_α as additional degrees of freedom, and vary them and the q_i independently.

But the Euler-Lagrange Eqs for λ_α are the constraint eqs. $f_\alpha = 0$.

Varying the q_i's gives,

$$\delta I = \int_1^2 dt \left(\sum_{i=1}^{n} \left(-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} \right) \delta q_i \right) = 0$$

Choose the λ_α so that m eqs. are satisfied $\forall\, \delta q_i$, the remaining variations δq_i are independent.

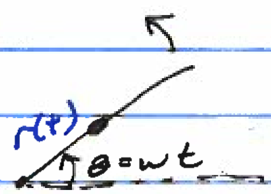
$\Rightarrow m + (n-m)$ eqs. of the form

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_k} \equiv Q_k$$

Generalized forces that enforce the constraint.

Example: Bead on rotating straight wire, constant rotational speed $\omega = \dot{\theta}$, no gravity.



$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

constraint: $\theta - \omega t = 0$

$$I = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \lambda(\theta - \omega t)\right]$$

$$\frac{\delta I}{\delta \lambda} = 0 \rightarrow \theta - \omega t = 0$$

$$\frac{\delta I}{\delta \theta} = 0 \rightarrow \frac{d}{dt}(m r^2 \dot{\theta}) - \lambda = 0 \rightarrow \lambda = \frac{d}{dt}(\underbrace{m r^2 \omega}_{\text{Angular momentum}})$$

$= N$ torque

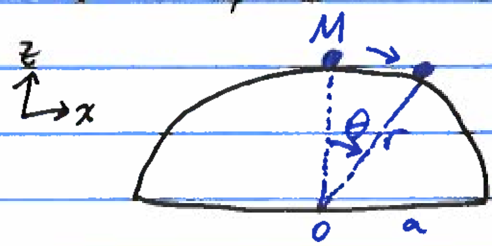
$$\frac{\delta I}{\delta r} = 0 \rightarrow \frac{d}{dt}(m \dot{r}) - m r \dot{\theta}^2 = 0 \rightarrow \frac{d}{dt}(m\dot{r}) - m r \omega^2 = 0$$

Generalized force $Q_\theta = \lambda \frac{\partial}{\partial \theta}(\theta - \omega t) = \lambda = N$ torque

c.f. $Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta} = \vec{F} \cdot (r \hat{\theta}) = r F_\theta = N$ ✓

The generalized force is the torque required to keep the bead on the wire.

Example: A small mass M slides w/o friction down a hemisphere of radius a attached to Earth



M is initially displaced infinitessimally from the top of the hemisphere in the x-z plane.

Motion is in the x-z plane.

$L = \frac{1}{2}M(\dot{x}^2 + \dot{z}^2) - Mgz = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mg\,r\cos\theta$

Constraint: $r - a = 0$

Action $I = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mg\,r\cos\theta + \lambda(r-a)$

Equations of motion:

$\delta\lambda$: $r = a$

δr: $\frac{d}{dt}(M\dot{r}) - Mr\dot{\theta}^2 + Mg\cos\theta - \lambda = 0$

$\Rightarrow \quad -Ma\dot{\theta}^2 + Mg\cos\theta - \lambda = 0$

$\delta\theta$: $\frac{d}{dt}(Mr^2\dot{\theta}) - Mg\,r\sin\theta = 0$

$\Rightarrow \quad Ma^2\ddot{\theta} - Mga\sin\theta = 0$

solution: $\dot{\theta}^2 = -\frac{2g}{a}\cos\theta + \frac{2g}{a}$

so that $\dot{\theta} = 0$ when $\theta = 0$.

Then $\lambda = Mg(-2 + 3\cos\theta)$

Generalized force $Q_r = \lambda\frac{\partial f}{\partial r} = \lambda$

Recall that this is also $Q_r = \vec{F}\cdot\frac{\partial\vec{r}}{\partial r} = F_r$

This is the normal force constraining the mass to the surface.

We must have $F_r > 0 \rightarrow$ the mass leaves the surface when

$\boxed{\cos\theta = 2/3.}$