More Examples of Relativistic Reactions

Photons have both energy and momentum. Clearly the relations $E = mc^2\gamma$, $\vec{p} = m\vec{v}\gamma$ don't apply. $m \to 0$, while $\gamma \to \infty$ as $v \to c$.

The kinematical relation $E^2 = p^2c^2 + m^2c^4$ is smooth in the limit $m \to 0$, so for photons we have the relation

$\boxed{E_\gamma = P_\gamma c}$, where the γ subscript indicates that the relation is only valid for photons, denoted by γ.

Example:

Consider the reaction $\gamma + p \to \Sigma^0 + K^+$.

photon ↗ ↖ proton

Question: what is the threshold photon energy in the proton's rest frame (i.e. the lab frame)?

Lab frame: $E = P_\gamma c + m_p c^2$, $\vec{p} = \vec{P}_\gamma$

total $P^\mu_{\text{initial}} = \left(P_\gamma + m_p c, \vec{P}_\gamma\right)$

$$\boxed{(P^\mu P_\mu)_{\text{initial}} = (P_\gamma + m_p c)^2 - P_\gamma^2 = m_p^2c^2 + 2m_p P_\gamma c}$$

At threshold in the COM frame, the Σ^0 and K^+ are at rest.

COM frame: $P^\mu_{\text{final}} = \left((m_\Sigma + m_K)c, \vec{0}\right)$

$$\boxed{(P_\mu P^\mu)_{final} = (m_K + m_\Sigma)^2 c^2}$$

Because $(P^\mu P_\mu)$ is Lorentz invariant, it takes the same value in any reference frame.
Because P^μ is conserved, $(P^\mu P_\mu)_{initial} = (P^\mu P_\mu)_{final}$.

Hence, $m_p^2 c^2 + 2 m_p p_\gamma c = (m_K + m_\Sigma)^2 c^2$

Solving for $p_\gamma c$, the threshold photon energy:

$$\boxed{p_\gamma c = \frac{(m_K + m_\Sigma)^2 c^2 - m_p^2 c^2}{2 m_p}}$$

This is what we were looking for.

plugging in $\begin{cases} m_{\Sigma^0} = 1193 \text{ MeV}/c^2 \\ m_{K^+} = 494 \text{ MeV}/c^2 \\ m_p = 938 \text{ MeV}/c^2 \end{cases}$

gives $\boxed{p_\gamma c = 1050 \text{ MeV}}$,

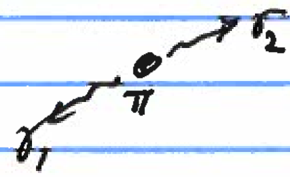
which is a little more than the proton's rest energy, and more than the difference between final and initial rest energies.

<u>Example</u>: Consider the decay $\pi^0 \to \gamma\gamma$.

<u>Question</u>: Suppose in the lab frame the pion has speed V. What can be said about the momenta of the photons?

The analysis is easy in the COM frame:

$$E_\pi = m_\pi c^2, \quad \vec{p}_\pi = \vec{0}.$$



COM frame.

Hence, in the COM frame $\vec{p}_{\gamma_1} = -\vec{p}_{\gamma_2}$, $2p_\gamma c = m_\pi c^2$

$$\rightsquigarrow |\vec{p}_{\gamma_1}| = |\vec{p}_{\gamma_2}| = \frac{m_\pi c}{2}$$

$\overset{S'}{\text{Lab}}$ $\overset{S}{\text{COM}}$

Suppose in the lab frame the pion is moving in the x-direction. Boosting the photon 4-momentum by velocity $V\hat{x}$ gives:

$$\begin{pmatrix} E'_\gamma/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-V^2/c^2}} & \frac{+V/c}{\sqrt{1-V^2/c^2}} & 0 & 0 \\ \frac{+V/c}{\sqrt{1-V^2/c^2}} & \frac{1}{\sqrt{1-V^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{m_\pi c}{2} \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\to E'_\gamma = \frac{1}{\sqrt{1-V^2/c^2}}\left(\frac{m_\pi c^2}{2} + p_x V\right)$$

$$p_x' = \frac{\left(\frac{m_\pi V}{2} + p_x\right)}{\sqrt{1-V^2/c^2}}, \quad p_y' = p_y, \quad p_z' = p_z$$

Note that $$p_\gamma'^2 = \frac{\left(\frac{m_\pi v}{2} + p_x\right)^2}{1 - v^2/c^2} + p_y^2 + p_z^2$$

$$= \frac{\left(\frac{m_\pi v}{2}\right)^2 + \frac{2 m_\pi v p_x}{2} + p_x^2 + p_z^2(1 - v^2/c^2) + p_y^2(1 - v^2/c^2)}{1 - v^2/c^2}$$

Using $p_x^2 + p_y^2 + p_z^2 = \left(\frac{m_\pi c}{2}\right)^2$:

$$p_\gamma'^2 = \frac{\left(\frac{m_\pi v}{2}\right)^2 + \frac{2 m_\pi v p_x}{2} + \left(\frac{m_\pi c}{2}\right)^2 (1 - v^2/c^2) + \frac{p_x^2 v^2}{c^2}}{1 - v^2/c^2}$$

$$= \left(\left(\frac{m_\pi c}{2}\right)^2 + \frac{2 m_\pi v p_x}{2} + \left(\frac{p_x v}{c}\right)^2\right) \frac{1}{1 - v^2/c^2}$$

$$= \left(\frac{m_\pi c}{2} + \frac{p_x v}{c}\right)^2 \cdot \frac{1}{1 - v^2/c^2}$$

$$\Rightarrow \boxed{p_\gamma' = \left(\frac{m_\pi c}{2} + \frac{p_x v}{c}\right) \frac{1}{\sqrt{1 - v^2/c^2}}}$$

And we have $E_\gamma' = p_\gamma' c$, as in the COM frame.

Lagrangian Formulation of Relativistic Mechanics

We have already observed that the proper time elapsed along an inertial trajectory connecting two events is maximum compared to other trajectories.

Hence, for the free relativistic particle we have Hamilton's principle,

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = 0, \text{ with}$$

$$L = -mc^2 \sqrt{1 - \dot{\vec{x}}^2/c^2},$$

so that

$$S = -mc^2 \underset{\nwarrow \text{ proper time along trajectory.}}{\tau}$$

A generalization to a particle under a conservative force $V(\vec{x})$ could be:

$$L = -mc^2 \sqrt{1 - \dot{\vec{x}}^2/c^2} - V(\vec{x}).$$

The Lagrange eqs are:

$$\frac{d}{dt}\underbrace{\left(\frac{m\dot{\vec{x}}}{\sqrt{1-\dot{\vec{x}}^2/c^2}} \right)}_{= m\vec{v}\gamma \text{ relativistic momentum}} = -\nabla V$$

The canonical momentum is the same as the mechanical momentum in this case: $\vec{p} = m\vec{v}\gamma$.

The energy function is $h = \vec{p}\cdot\dot{\vec{x}} - L$

$$h = \frac{m\dot{\vec{x}}^2}{\sqrt{1-\dot{\vec{x}}^2/c^2}} + mc^2\sqrt{1-\dot{\vec{x}}^2/c^2} + V(\vec{x})$$

$$= \frac{mc^2\left(\dot{\vec{x}}^2/c^2 + \left(1-\dot{\vec{x}}^2/c^2\right)\right)}{\sqrt{1-\dot{\vec{x}}^2/c^2}} + V(\vec{x})$$

$$= \underbrace{\frac{mc^2}{\sqrt{1-\dot{\vec{x}}^2/c^2}}}_{\text{rest energy + kinetic energy}} + \underbrace{V(\vec{x})}_{\text{potential energy.}}$$

A more covariant formalism with additional subtleties was described in the lecture on the Hamiltonian formalism