Special Relativity

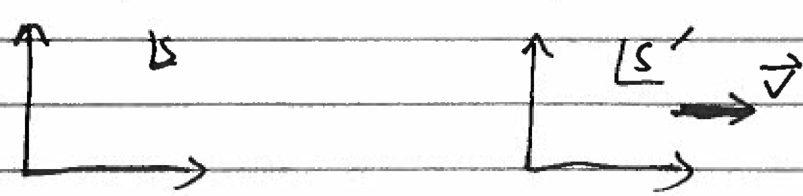
When Maxwell realized that the equations of electromagnetism predict self-propagating wavelike solutions with a universal speed $c = 1/\sqrt{\epsilon_0 \mu_0}$, physics was faced with a choice: Either c is the speed of light in a preferred frame, in which case Maxwell's equations are only valid in that frame; or else the relation between velocities as measured in different inertial frames had to be modified.

Special relativity is Einstein's replacement of Newtonian mechanics with new laws of mechanics consistent with the universality of the speed of light in all inertial frames.

Two postulates of Special Relativity

1. The laws of physics are the same to all inertial observers.
2. The speed of light is the same to all inertial observers.

It will be helpful to consider two inertial frames related by uniform motion by some velocity $\vec{v}$:



Newtonian relation between spacetime coordinates
$\equiv$ Galilean transformations.

$$\left.\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}\right\} \qquad \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \vec{F}' = \frac{d\vec{p}\,'}{dt'} = \frac{d\vec{p}\,'}{dt}$$

Newtonian velocity addition: If $\vec{u}$ = velocity of a particle in S
and $\vec{u}\,'$ = velocity of the particle in S',
$\vec{u}\,' = \vec{u} - \vec{v}$.

Newtonian velocity addition does not allow a constant speed independent of inertial frame.

The transformation between time intervals and spatial intervals in special relativity leaves invariant the interval

$(\Delta s)^2 = c^2 (\Delta t)^2 - (\Delta \vec{x})^2$ — invariant interval.

Infinitesimal form: $(ds)^2 = c^2 (dt)^2 - (d\vec{x})^2$.
(Analogy: Rotations leave invariant $(d\vec{x})^2$)
Lightlike trajectory: $\left|\frac{dx}{dt}\right| = c \Rightarrow (ds)^2 = 0$.

The proper time along a trajectory is the time elapsed according to a clock at rest relative to the moving body.

Suppose an object is at rest in the frame S'.

we have $\quad c^2(dt')^2 - (\underbrace{\vec{0}}_{(d\vec{x}')^2})^2 = c^2(dt)^2 - (d\vec{x})^2$

or, $\quad \underbrace{c^2(d\tau)^2}_{\text{proper time } \tau} = c^2(dt)^2 - v^2(dt)^2$

$$= c^2(dt)^2(1 - v^2/c^2)$$

$$\equiv c^2(dt)^2(1-\beta^2)$$

where $\beta \equiv v/c$.

$\Rightarrow$ Time elapsed in frame S is related to time elapsed in frame S' by

$$\boxed{dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}}} \geq d\tau \rightarrow \text{Time Dilation}$$

"Moving clocks run slow."

Define $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} \geq 1$

The transformation between times and positions in frames S and S' is given by the Lorentz boost, which preserves $(ds)^2$:

$$\begin{cases} ct' = \dfrac{ct - \beta x}{\sqrt{1-\beta^2}} = \gamma(ct - \beta x) \\[2ex] x' = \dfrac{x - \beta ct}{\sqrt{1-\beta^2}} = \gamma(x - \beta ct) \\[2ex] y' = y \\ z' = z \end{cases}$$

These are the Lorentz transformations relating the two frames.

As a linear transformation in matrix notation:

$$\underbrace{\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}}_{x'^{\mu}} = \underbrace{\sum_{\nu=0}^{3}}_{} \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{L^{\mu}{}_{\nu}} \underbrace{\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}}_{x^{\nu}}$$

$x^0 = ct$, $\quad x^1 = (\vec{x})_1 = x$, $\quad x^2 = (\vec{x})_2 = y$, $\quad x^3 = (\vec{x})_3 = z$

Einstein summation convention: $x'^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$

sum on repeated indices is implied.

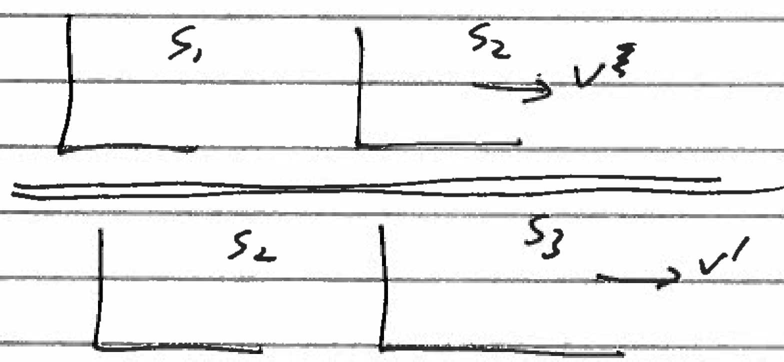
More generally, if S' and S are related by $\vec{\beta} = \vec{v}/c$,

$$\begin{cases} (ct') = \gamma(ct - \vec{\beta}\cdot\vec{r}) \\ \vec{r}\,' = \vec{r} + \dfrac{(\vec{\beta}\cdot\vec{r})\vec{\beta}(\gamma-1)}{\beta^2} - \vec{\beta}\gamma c t \end{cases}$$

<u>Exercise</u>: Write the components of the Lorentz transformation matrix $L^{\mu}{}_{\nu}$ in the general case.

Velocity addition along the same axis

Consider a succession of two Lorentz boosts in the x direction, so that there are 3 frames under consideration: S_1, S_2, S_3, with S_2 moving w/ v along the x-direction relative to S_1, and S_3 moving w/ v' along the x-direction relative to S_2.



The Lorentz transformation from S_1 to S_3 is given by the matrix

$$\underline{L}_{1\to 3} = \begin{bmatrix} \gamma' & -\gamma'\beta' & 0 & 0 \\ -\gamma'\beta' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \gamma\gamma'(1+\beta\beta') & -\gamma\gamma'(\beta+\beta') & 0 & 0 \\ -\gamma\gamma'(\beta+\beta') & \gamma\gamma'(1+\beta\beta') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Write $\underline{L}_{1\to3}$ as a Lorentz boost:

$$\underline{L}_{1\to3} = \begin{bmatrix} \gamma'' & -\gamma''\beta'' & 0 & 0 \\ -\gamma''\beta'' & \gamma'' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}} \quad \text{velocity addition}$$

<u>Thomas Precession</u>

The most general Lorentz transformation can be written as a product of a Lorentz boost and a rotation:

$$\underline{L} = \underbrace{\underline{R}}_{\text{rotation}}\,\underbrace{\underline{L}_0}_{\text{boost}} \quad \text{or} \quad \underline{L} = \underbrace{\underline{L}_0'}_{\text{boost}}\,\underbrace{\underline{R}'}_{\text{rotation.}}$$

$\underline{L}_0$ and $\underline{R}$ don't generally commute

$\rightarrow$ $\underline{L}_0' \neq \underline{L}_0$ and $\underline{R}' \neq \underline{R}$ in general.

Consider a boost by v in the x-direction followed by a boost by v' in some other direction, with $v' \ll v$ and $v' \ll c$. Suppose $\vec{v}\,'$ is in the $x'y'$ plane of frame S_2.

Exercise: The product of the two Lorentz transformations is

$$\underline{L}'' = \underline{L}'\underline{L} = \begin{bmatrix} \gamma\gamma' & -\gamma\gamma'\beta & -\gamma'\beta'_{y'} & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\gamma\gamma'\beta'_{y'} & \gamma\beta\gamma'\beta'_{y'} & \gamma' & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $\gamma' \approx 1$ (because $\beta' \ll 1$):

$$\underline{L}'' \approx \begin{bmatrix} \gamma & -\gamma\beta & -\beta'_{y'} & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\gamma\beta'_{y'} & \gamma\beta\beta'_{y'} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is of the form $\underline{R}\,\underline{L}_{1\to3}$, where $\underline{R}$ is a rotation in the xy plane, and we read off the net boost velocity by comparison with the expression for the general Lorentz transformation with boost in the xy plane:
with

$$\underline{L}'' = \underline{R}\,\underline{L}_{1\to3}$$

$$\underline{L}_{1\to3} = \begin{bmatrix} \gamma'' & -\gamma''\beta_x'' & -\gamma''\beta_y'' & 0 \\ -\gamma''\beta_x'' & 1+(\gamma''-1)\frac{\beta_x''^2}{\beta''^2} & (\gamma''-1)\frac{\beta_x''\beta_y''}{\beta''^2} & 0 \\ -\gamma''\beta_y'' & (\gamma''-1)\frac{\beta_x''\beta_y''}{\beta''^2} & 1+(\gamma''-1)\frac{\beta_y''^2}{\beta''^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\beta'' \approx \beta_x''$

unchanged by multiplication
by B on left.

From the first row of $L'' = B L_{1\to 3}$:

$$\left\{\begin{array}{l} \beta_{x''}' \approx \beta_x'' = \beta \\ \beta_{y''}'' \approx \beta_y'' = \dfrac{\beta_y'}{\gamma} \\ \beta''^2 \approx \beta^2 \\ \gamma'' \approx \gamma \end{array}\right\}$$

The rotation matrix R induced from the rotation from S_1 to $S_{2,3}$:

Exercise:

$$R = L'' L_{1\to 3}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (\gamma-1)\frac{\beta_y''}{\beta} & 0 \\ 0 & -(\gamma-1)\frac{\beta_y''}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Rotation by small angle

$$\Delta\Omega \approx (\gamma-1)\frac{\beta_y''}{\beta} = \beta_y''\beta\left(\frac{\gamma-1}{\beta^2}\right)$$

Suppose a particle has a fixed spin $\vec{S}$ in its rest frame. As the particle changes its direction, we can consider successive boosts from the lab frame S_1 to to the particle's instantaneous rest frame S_2 at time t, to the " " " " S_3 at time $t+\Delta t$.

in x-direction ↓ ← in y-direction

$$\Rightarrow \Delta\vec{\Omega} = -(\gamma-1)\frac{\vec{v}\times\Delta\vec{v}}{v^2}$$

→ Precession observed in lab frame

$$\vec{\omega} = \frac{d\vec{\Omega}}{dt} = -(\gamma - 1)\,\frac{\vec{v} \times \left(\frac{d\vec{v}}{dt}\right)}{v^2}$$

← acceleration as seen in S_1

Using $\gamma \approx 1 + \frac{1}{2}\beta^2$ if $\beta \ll 1$

$$\rightarrow \boxed{\vec{\omega} \approx \frac{1}{2c^2}\left(\frac{d\vec{v}}{dt} \times \vec{v}\right)}$$

Thomas precession