Hamilton's Equations of Motion - Goldstein Ch. 8

The Hamiltonian formulation of mechanics is an alternative to the Lagrangian formulation. Whereas the Lagrangian formulation makes direct contact with the functional integral (i.e. path integral) representation of quantum mechanics, the Hamiltonian formulation is immediately recognizable from the canonical Heisenberg and Schrödinger formulations of quantum mechanics.

From Lagrangians to Hamiltonians:

$L(q_i, \dot{q}_i, t)$, $i \in \{1, \dots, n\}$

Assume the q_i are independent (no constraints).

The Hamiltonian description is in terms of $2n$ independent variables q_i and their conjugate momenta p_i,

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}$$

The set of $\{q_i, p_i\}$ are called canonical variables. The transformation from $\{q_i, \dot{q}_i\}$ to $\{q_i, p_i\}$ is a Legendre transformation.

Suppose a function $f(x, y)$ has differential

$$df = u\,dx + v\,dy \,, \quad \text{where}$$

$$u = \frac{\partial f}{\partial x} \,, \quad v = \frac{\partial f}{\partial y}$$

We can instead express the differentials in terms of du and dy:

Define $g = f - ux$

$$dg = df - u\,dx - x\,du$$
$$= (u\,dx + v\,dy) - u\,dx - x\,du$$
$$= v\,dy - x\,du$$

Here, $x = -\frac{\partial g}{\partial u}$, $v = \frac{\partial g}{\partial y}$

Rather than $f(x,y)$, consider $L(q_i, \dot{q}_i, t)$.

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

Euler-Lagrange

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \xrightarrow[\text{Lagrange}]{\text{Euler-}} \dot{p}_i = \frac{\partial L}{\partial q_i}$$

$$dL = \dot{p}_i\,dq_i + p_i\,d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

Define $\boxed{H(q_i, p_i, t) = \sum_i \dot{q}_i p_i - L(q_j, \dot{q}_j, t)}$

$$dH = \dot{q}_i\,dp_i - \dot{p}_i\,dq_i - \frac{\partial L}{\partial t} dt$$
$$= \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad -\dot{p}_i = \frac{\partial H}{\partial q_i} \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}}$$

Hamilton's Equations

Recall the energy function, which is conserved if $\frac{\partial L}{\partial t} = 0$:

$$h(q_i, \dot{q}_i, t) = p_i \dot{q}_i - L(q_j, \dot{q}_j, t) = E \text{ if } \frac{\partial L}{\partial t} = 0.$$

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$.

The energy fn. h is a function of $q_i, \dot{q}_i, t$.
The Hamiltonian $H(q_i, p_i, t)$ is a function of q_i, p_i, t.
Otherwise, Hamilton's equations don't apply.

To convert L to H requires inverting $p_i = \frac{\partial L}{\partial \dot{q}_i}$ to write $\dot{q}_i$ as a fn. of $\{q_j, p_j, t\}$.
The procedure fails if the equations are not invertible.

Example: Suppose $L = \frac{1}{2} m (\dot{r}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 + r^2 \dot{\theta}^2) - V(r, \theta, \phi)$

$$\begin{cases} p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \\ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \, \dot{\phi} \end{cases}$$

These equations are trivially inverted to give $\{\dot{r}, \dot{\theta}, \dot{\phi}\}$ in terms of $\{p_r, p_\theta, p_\phi, r, \theta, \phi\}$.

Then $H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2\theta} p_\phi^2\right) + V(r, \theta, \phi)$

Note! Don't confuse p_r, p_θ, p_ϕ with the $\hat{r}, \hat{\theta}, \hat{\phi}$ components of the linear momentum $\vec{p}$.

Example: Nonrelativistic particle of mass m and charge q moving in an electromagnetic field.

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - q\phi(\vec{r}) + q\vec{A}(\vec{r}) \cdot \dot{\vec{r}}$$

$$= \frac{1}{2} m \dot{x}_i \dot{x}_i - q\phi + qA_i \dot{x}_i$$

Canonical momenta: $p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i$

Invert: $\dot{x}_i = \frac{1}{m}(p_i - qA_i)$

Hamiltonian: $H = p_i \dot{x}_i - L$

$$= \frac{1}{2m}(p_i - qA_i)(p_i - qA_i) + q\phi$$

In these (Cartesian) coordinates, p_i are the components of a vector $\vec{p}$, so we can write

$$H = \frac{1}{2m}(\vec{p} - q\vec{A}) \cdot (\vec{p} - q\vec{A}) + q\phi$$

$$= \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi$$

Cyclic Coordinates

Suppose $L(q_i, \dot{q}_i, t)$ is independent of q_1 (but still depends on $\dot{q}_1$).

Then $p_1 = \frac{\partial L}{\partial \dot{q}_1}$ is conserved by Nöether's theorem:

$$\dot{p}_1 = 0$$

By Hamilton's equations, $\dot{p}_1 = -\frac{\partial H}{\partial q_1}$, so the Hamiltonian

is also independent of q_1.

Conversely, if H does not depend on q_1, then
$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0$, so p_1 is conserved.

Conservation of Energy:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = -\dot{p}_i \dot{q}_i + \dot{q}_i \dot{p}_i + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t} \quad \text{by Hamilton's equations}$$

$$= -\frac{\partial L}{\partial t}$$

Hence, if H is not an explicit function of t, then $H = E$ is a constant of the motion.

What if $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is not invertible, i.e.

what if it is not possible to find $\dot{q}_i(\{q_j, p_j, t\})$?

Example: $L = -mc\sqrt{\dot{x}_0^2 - \dot{x}_1^2}$

$$P_0 \equiv \frac{\partial L}{\partial \dot{x}_0} = \frac{-mc\dot{x}_0}{\sqrt{\dot{x}_0^2 - \dot{x}_1^2}}$$

$$P_1 \equiv \frac{\partial L}{\partial \dot{x}_1} = \frac{mc\dot{x}_1}{\sqrt{\dot{x}_0^2 - \dot{x}_1^2}}$$

Note that $P_0^2 - P_1^2 = m^2c^2$, so P_0 and P_1 are not independent. $\Rightarrow$ Dirac's Constraint Formalism

Namely:

The Hamiltonian is $H = P_0\dot{x}_0 + P_1\dot{x}_1 - L = 0.$

Treating $x_0(t)$ and $x_1(t)$ as independent generalized coordinates gives a Hamiltonian independent of (x_0, x_1, P_0, P_1, t). Clearly something is amiss.

The Euler-Lagrange equations are

$$\begin{cases} \frac{d}{dt}\left(\frac{-mc\dot{x}_0}{\sqrt{\dot{x}_0^2 - \dot{x}_1^2}}\right) = 0 \\ \frac{d}{dt}\left(\frac{mc\dot{x}_1}{\sqrt{\dot{x}_0^2 - \dot{x}_1^2}}\right) = 0 \end{cases}$$

The first equation, for $x_0(t)$, can be written,

$$\frac{d}{dt}\left(mc\sqrt{1+\frac{\dot{x}_1^{\,2}}{\dot{x}_0^{\,2}-\dot{x}_1^{\,2}}}\right)=0$$

$$= \frac{+1}{\sqrt{1+\frac{\dot{x}_1^{\,2}}{\dot{x}_0^{\,2}-\dot{x}_1^{\,2}}}}\ \frac{\dot{x}_1}{\sqrt{\dot{x}_0^{\,2}-\dot{x}_1^{\,2}}}\ \frac{d}{dt}\left(\frac{mc\,\dot{x}_1}{\sqrt{\dot{x}_0^{\,2}-\dot{x}_1^{\,2}}}\right)=0$$

$$= \frac{\dot{x}_1}{\dot{x}_0}\ \frac{d}{dt}\left(\frac{mc\,\dot{x}_1}{\sqrt{\dot{x}_0^{\,2}-\dot{x}_1^{\,2}}}\right)=0$$

But then the first equation is satisfied whenever the equation for $x_1(t)$ is satisfied.

Any monotonic function $x_0(t)$ gives an acceptable solution $\longrightarrow$ the dynamics does not constrain $x_0(t)$.

For example, we can choose $x_0(t) = ct$.

Then $\dfrac{d}{dt}\left(\dfrac{m\,\dot{x}_1}{\sqrt{1-\dot{x}_1^{\,2}/c^2}}\right)=0$

Gauge invariance

Suppose
$$\begin{cases} x_0(t) \to x_0(t(t')) \equiv \tilde{x}_0(t') \\ x_1(t) \to x_1(t(t')) \equiv \tilde{x}_1(t') \end{cases}$$

The action
$$S = \int_{t_1}^{t_2} \left(-mc\sqrt{\dot{x}_0^{\,2} - \dot{x}_1^{\,2}}\right) dt$$
$$= \int_{t_1}^{t_2} \left(-mc\sqrt{\left(\frac{d\tilde{x}_0}{dt'}\right)^2 \left(\frac{dt'}{dt}\right)^2 - \left(\frac{d\tilde{x}_1}{dt'}\right)^2 \left(\frac{dt'}{dt}\right)^2}\right) dt$$
$$= \int_{t_1'}^{t_2'} \left(-mc\sqrt{\left(\frac{d\tilde{x}_0}{dt'}\right)^2 - \left(\frac{d\tilde{x}_1}{dt'}\right)^2}\right) dt'$$

The action takes the same form for any reparametrization of the time parameter t.

In order to provide meaning to the dynamics, it is necessary to specify, for example, $x_0(t)$. Only then is $x_1(t)$ a meaningful description of the time-dependence of x_1.

The redundancy of the description of the dynamics as a result of the independence of the action on local transformations of the generalized coordinates is an example of gauge invariance.