Forced Vibrations

Consider a generalized force F_i acting on coordinate η_i.

The normal coordinates were related to η_i by

$$\eta_i = A_{ij} \zeta_j , \quad \text{where } A_{ij} = (a_j)_i .$$

The generalized force acting on normal coordinate ζ_j is then

$$Q_j = F_i \frac{\partial \eta_i}{\partial \zeta_j} = F_i A_{ij}$$

The equations of motion for the normal coords. are:

$$\ddot{\zeta}_i + \omega_i^2 \zeta_i = Q_i(t)$$

Consider a sinusoidal driving force, e.g. from a pressure wave or monochromatic light.

Suppose

$$Q_i = Q_{0i} \cos(\omega t + \delta_i)$$

ω: angular frequency of external driving force.

$$\ddot{\zeta}_i + \omega_i^2 \zeta_i = Q_{0i} \cos(\omega t + \delta_i)$$

Particular sol'n: $\zeta_i = B_i \cos(\omega t + \delta_i)$

$$\left(-B_i \omega^2 + B_i \omega_i^2\right) \cos(\omega t + \delta_i) = Q_{0i} \cos(\omega t + \delta_i)$$

$$\rightarrow \boxed{B_i = \frac{Q_{0i}}{\omega_i^2 - \omega^2}}$$

Complete motion:

$$\eta_j = A_{ji} \zeta_i = \frac{A_{ji} Q_{0i} \cos(\omega t + \delta_i)}{\omega_i^2 - \omega^2}$$

Normal oscillations now occur at the driving frequency ω.

As ω approaches one of the resonance frequencies ω_i, the amplitude of that mode becomes large.

In reality, the assumption of small disturbance from equilibrium breaks down as $\omega \to \omega_i$, and the divergence in the amplitude is fictitious.

Note that the sign of ζ_i changes as ω passes through ω_i. There is a phase change of π in going through the resonance.

Dissipative Forces

Suppose there is a dissipative force that depends on velocity,

$$F_i = -\frac{\partial \mathcal{F}}{\partial \dot{\eta}_i}, \text{ with}$$

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{ij} \dot{\eta}_i \dot{\eta}_j \quad \text{for some constants } \mathcal{F}_{ij}.$$

Lagrangian: $L = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j - \frac{1}{2} V_{ij} \eta_i \eta_j$

Lagrange Eqs: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_i}\right) - \frac{\partial L}{\partial \eta_i} = F_i = -\mathcal{F}_{ij} \dot{\eta}_j$

$$\Rightarrow \quad \boxed{T_{ij} \ddot{\eta}_j + V_{ij} \eta_j + \mathcal{F}_{ij} \dot{\eta}_j = 0}$$

Suppose F_i is proportional to particle's mass. Then $\mathcal{F}_{ij}$ and T_{ij} are simultaneously diagonal.

In that case, the equations of motion are decoupled in normal coordinates:

$$\ddot{\zeta}_i + \mathcal{F}_i \dot{\zeta}_i + \omega_i^2 \zeta_i = 0$$

Diagonal elements of diagonalized $\mathcal{F}_{ij}$, $\mathcal{F}_i \geq 0$.

Ansatz: $\zeta_i = C_i e^{-i\omega_i' t}$

Plugging the ansatz into the equations of motion:

$$\omega_i'^2 + i\omega_i' \mathcal{F}_i - \omega_i^2 = 0. \quad \text{(no sum)}$$

Solutions: $$\omega_i' = \frac{-i\mathcal{F}_i \pm \sqrt{4\omega_i^2 - \mathcal{F}_i^2}}{2}$$

ω_i' is complex — not a pure oscillation.

$$\boxed{\zeta_i = C_i\, e^{-\mathcal{F}_i t/2}\, e^{-i\sqrt{\omega_i^2 - \frac{\mathcal{F}_i^2}{4}}\, t}}$$

exponential damping.

oscillations (for small $\mathcal{F}_i$)

If $\mathcal{F}_i \ll \omega_i$, then

$$\boxed{\zeta_i \approx C_i\, e^{-\mathcal{F}_i t/2}\, e^{-i\omega_i t}}$$

Forced sinusoidal oscillations with dissipative forces

Driving force $F_j = F_{0j}\, e^{-i\omega t}$

$$V_{ij}\eta_j + \mathcal{F}_{ij}\dot{\eta}_j + T_{ij}\ddot{\eta}_j = F_{0i}\, e^{-i\omega t}$$

Particular Solution: $\eta_j = A_j\, e^{-i\omega t}$ ansatz

$$\rightarrow \left(V_{ij} - i\omega\mathcal{F}_{ij} - \omega^2 T_{ij}\right) A_j - F_{0i} = 0$$

Define $D(\omega) = \det\left(V_{ij} - i\omega F_{ij} - \omega^2 T_{ij}\right)$,

and $D_j(\omega) = D(\omega)$ with j^{th} column of the matrix $V_{ij} - i\omega F_{ij} - \omega^2 T_{ij}$ replaced by $F_{01}, \dots, F_{0n}$.

$D(\omega)$ can be represented as

$$D(\omega) = G\,(\omega-\omega_1)(\omega-\omega_2)\cdots(\omega-\omega_n)(\omega+\omega_1^*)(\omega+\omega_2^*)\cdots(\omega+\omega_n^*)$$

↖ constant.

If ω_i is a root of $D(\omega)$, so is $-\omega_i^*$.

Write $\omega = 2\pi\nu$, $\omega_i = 2\pi\nu_i - iK_i$

$$\Rightarrow D(\omega) = G\prod_{i=1}^{n}\left[2\pi(\nu-\nu_i) + iK_i\right]\left[2\pi(\nu+\nu_i) + iK_i\right]$$

$$|D(\omega)|^2 = D(\omega)^* D(\omega)$$

$$= GG^*\prod_{i=1}^{n}\left[4\pi^2(\nu-\nu_i)^2 + K_i^2\right]\left[4\pi^2(\nu+\nu_i)^2 + K_i^2\right]$$

Cramer's rule: Sol'n to $(V_{ij} - i\omega F_{ij} - \omega^2 T_{ij})A_j = F_{0i}$

is $A_j = \dfrac{D_j(\omega)}{D(\omega)}$

As frequency $\nu \to \nu_i$, $D(\omega) \to$ small $\Rightarrow$ Resonance.