The Euler Equations of Rigid Body Motion

In terms of the inertia tensor $\underline{I}$ and the angular velocity $\vec{\omega}$, the kinetic energy of rotation about the axis specified by $\vec{\omega}$ is

$$T = \frac{\vec{\omega} \cdot \underline{I} \vec{\omega}}{2} = \frac{\omega^2}{2} \hat{n} \cdot \underline{I} \hat{n} ,$$

or $$\boxed{T = \tfrac{1}{2} I \omega^2},$$

where $I = \hat{n} \cdot \underline{I} \hat{n} = \sum_a m_a \left[r_a^2 - (\vec{r}_a \cdot \hat{n})^2 \right]$ is the moment of inertia about the axis of rotation. Here we have assumed that the origin of the coordinate axes lies along the axis of rotation.

If we choose the axis of rotation to be through the center of mass, then the kinetic energy decomposes as

$$T = \underbrace{\tfrac{1}{2} M v^2}_{\text{translational}} + \underbrace{\tfrac{1}{2} I \omega^2}_{\text{rotational}} .$$

Suppose the Lagrangian separates into a part depending on only the translational generalized coordinates and a part depending only on the orientational generalized coordinates,

$$L = \underbrace{L_c(q_c, \dot{q}_c)}_{\text{translational}} + \underbrace{L_s(q_s, \dot{q}_s)}_{\text{rotational (body)}}$$

Then we can consider the rotational motion separately from the translational motion.

Consider an inertial frame whose origin is at the fixed point of the rigid body, or a system of space axes w/ origin at the center of mass.

Then $$\left(\frac{d\vec{L}}{dt}\right)_s = \vec{N} \quad \text{(torque)}$$

$$= \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L}$$

$\uparrow$ body

In terms of the body frame, we have

$$\frac{dL_i}{dt} + \epsilon_{ijk}\,\omega_j L_k = N_i$$

Suppose we take the body axes to be the principal axes relative to the origin, so that

$$L_i = I_i \omega_i. \quad \text{(not summed over } i\text{)}$$

Then,

$$I_i \frac{d\omega_i}{dt} + \sum_{jk} \epsilon_{ijk}\,\omega_j (I_k \omega_k) = N_i \quad \text{(not summed over } i\text{)}$$

$$\Rightarrow \begin{cases} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = N_1 \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = N_2 \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = N_3 \end{cases}$$

These are Euler's equations of motion.

Torque-free Rigid Body Motion

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3)$$

$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1)$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2)$$

Define the vector $\vec{\rho} = \dfrac{\vec{\omega}}{\sqrt{2T}} = \dfrac{\hat{n}}{\sqrt{I}}$, $I = \hat{n} \cdot (\underline{\underline{I}}\, \hat{n})$

T: rotational kinetic energy.

$$1 = \sum_{ij} I_{ij} \rho_i \rho_j = I_1 \rho_1'^2 + I_2 \rho_2'^2 + I_3 \rho_3'^2$$

in principal axis frame.

— Equation for an ellipsoid called the <u>inertial ellipsoid</u>.

Define $F(\vec{\rho}) \equiv \vec{\rho} \cdot (\underline{\underline{I}}\, \vec{\rho})$,

with $\vec{\rho}$ allowed to vary from its definition in terms of $\vec{\omega}$,
$F(\vec{\rho}) = 1$ is the inertial ellipsoid.

$$\nabla_\rho F = 2 \underline{\underline{I}} \cdot \rho = \frac{2 \underline{\underline{I}} \cdot \vec{\omega}}{\sqrt{2T}} \quad \text{on the inertial ellipsoid.}$$

gradient of F with respect to ρ

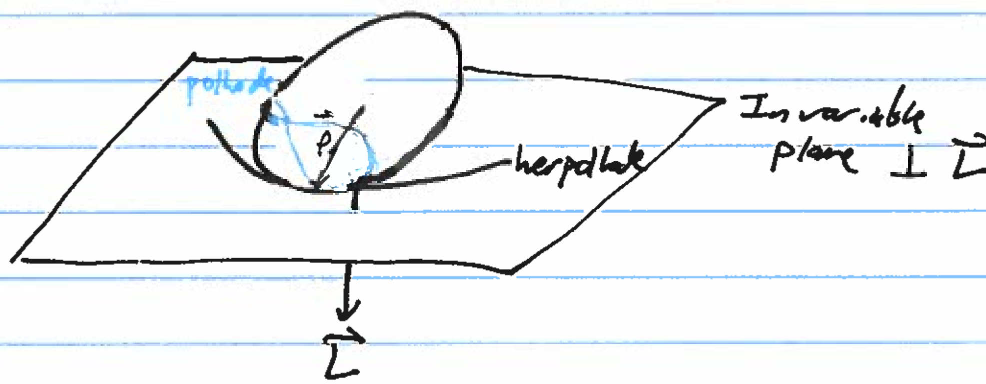
$$\nabla_\rho F = \frac{2 I \vec{\omega}}{\sqrt{2T}} = \sqrt{\frac{2}{T}}\,\vec{L}$$

Hence, $\vec{\omega}$ moves such that the normal to the inertia ellipsoid is in the direction of $\vec{L}$.

The distance from the origin of the ellipsoid to the plane tangent to it at $\vec{\rho}$ is

$$\vec{\rho}\cdot\left(\frac{\vec{L}}{L}\right) = \frac{\vec{\omega}\cdot\vec{L}}{\sqrt{2T}\,L} = \underbrace{\left(\frac{\vec{\omega}\cdot\vec{L}}{2}\right)}_{T}\cdot\sqrt{\frac{2}{T}}\,\frac{1}{L}$$

$$= \frac{\sqrt{2T}}{L} = \text{constant} \quad (\text{no forces, torques})$$



$\vec{\rho} = \frac{\vec{\omega}}{\sqrt{2T}}$ is momentarily at rest $\rightarrow$ The inertia ellipsoid rolls without slipping on the plane $\perp \vec{L}$ and tangent to the ellipsoid, called the <u>invariable plane</u>.

The point of contact of the ellipsoid with the invariable plane traces out the <u>polhode</u> on the ellipsoid and the <u>herpolhode</u> on the invariable plane.

Another ellipsoid can be constructed as

$$T = \frac{L_x^2}{2I_1} + \frac{L_y^2}{2I_2} + \frac{L_z^2}{2I_3}$$ in the principal axis frame.

Assume $I_3 \le I_2 \le I_1$.

$$\boxed{\frac{L_x^2}{2TI_1} + \frac{L_y^2}{2TI_2} + \frac{L_z^2}{2TI_3} = 1.}$$

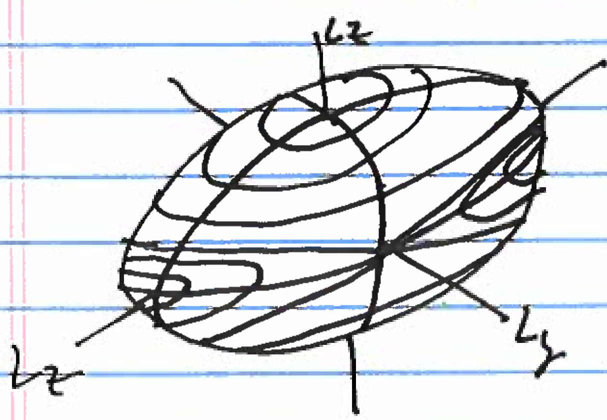
Binet ellipsoid.
= kinetic energy ellipsoid.

By conservation of $\vec{L}$, we also have

$$\boxed{\frac{L_x^2 + L_y^2 + L_z^2}{L^2} = 1}$$

$$\Rightarrow \boxed{\frac{L_x^2}{2TI_1} + \frac{L_y^2}{2TI_2} + \frac{L_z^2}{2TI_3} = \frac{L_x^2 + L_y^2 + L_z^2}{L^2}.}$$

Solution if $\sqrt{2TI_3} < L < \sqrt{2TI_1}$



Intersections of Binet (kinetic energy) ellipsoid and angular momentum sphere = possible paths of $\vec{L}$ in body frame.

Steady rotation ($\vec{L}$ fixed) is only possible through one of the principal axes:

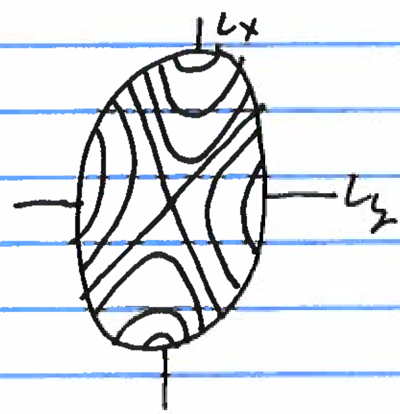
$$\omega_1 \omega_2 (I_1 - I_2) = \omega_2 \omega_3 (I_2 - I_3) = \omega_3 \omega_1 (I_3 - I_1) = 0$$

$\rightarrow$ At least two of $\omega_1, \omega_2, \omega_3$ must be zero.

<u>Example</u>: Steady motion about L_z axis $\Rightarrow L^2 = 2TI_3$.
Deviations from this condition are stable: small ellipse about the L_z axis.

Similarly for steady motion about L_x axis.

<u>However</u>, steady motion about the middle principal axis L_y is unstable, and will tend to deviate far from the L_y axis.



Paths of $\vec{L}$, side view of Binet ellipsoid from L_z axis.

Suppose $I_1 = I_2$.

$$\begin{cases} I_1 \dot{\omega}_1 = (I_1 - I_3)\,\omega_3 \omega_2 \\ I_1 \dot{\omega}_2 = (I_3 - I_1)\,\omega_3 \omega_1 \\ I_3 \dot{\omega}_3 = 0 \quad \longrightarrow \quad \omega_3 = \text{constant.} \end{cases}$$

$$\dot{\omega}_1 = -\Omega \omega_2, \quad \dot{\omega}_2 = \Omega \omega_1, \quad \Omega \equiv \frac{I_3 - I_1}{I_1}\,\omega_3$$

$$\ddot{\omega}_1 = -\Omega \dot{\omega}_2 = -\Omega^2 \omega_1$$

$$\rightarrow \omega_1 = A\cos(\Omega t + \alpha)$$

$$\dot{\omega}_2 = \Omega\left(A\cos(\Omega t + \alpha)\right)$$

$$\rightarrow \omega_2 = A\sin(\Omega t + \alpha)$$

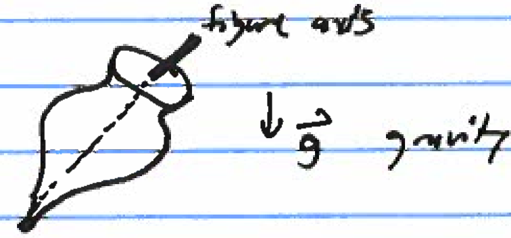
$\rightarrow$ $\omega_1 \hat{i} + \omega_2 \hat{j}$ has constant magnitude, precesses around z axis w/ angular frequency Ω.

Earth: $\frac{I_3 - I_1}{I_1} = 0.00327$

$$\Omega = \frac{\omega_3}{305.81} \approx \frac{|\vec{\omega}|}{305.81} \sim \text{once/ten months.}$$

– precession of Earth's axis

Symmetrical top with one point fixed



Motion of figure axis :
precession about vertical
+ nutation

For more detail : Goldstein 5.7