The Coriolis Effect, Fictitious forces

We have derived the transformation between time derivatives of vectors in the space frame and in the rotating (body) frame.

The mnemonic is:

$$\boxed{\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \vec{\omega}\times}$$

Defining $\vec{v}_s = \left(\frac{d\vec{r}}{dt}\right)_s$ the velocity of a particle relative to the space frame, and $\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_r$ the velocity relative to the rotating frame.

$$\boxed{\vec{v}_s = \vec{v}_r + \vec{\omega}\times\vec{r}}$$

Taking a second derivative, we have (with $\vec{\omega}$ constant):

$$\vec{a}_s \equiv \frac{d\vec{v}_s}{dt} = \left(\frac{d\vec{v}_s}{dt}\right)_r + \vec{\omega}\times\vec{v}_s$$

$$= \frac{d}{dt}\left(\vec{v}_r + \vec{\omega}\times\vec{r}\right)_r + \vec{\omega}\times\left(\vec{v}_r + \vec{\omega}\times\vec{r}\right)$$

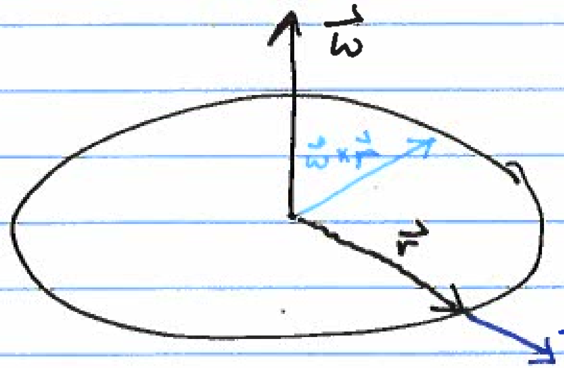
$$= \left(\frac{d\vec{v}_r}{dt}\right)_r + \left(\vec{\omega}\times\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\vec{v}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\boxed{\vec{a}_s = \vec{a}_r + 2(\vec{\omega}\times\vec{v}_r) + \vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

To an observer in the rotating system, it is as if the force $\vec{F} = m\vec{a}_s$ is replaced by the effective force

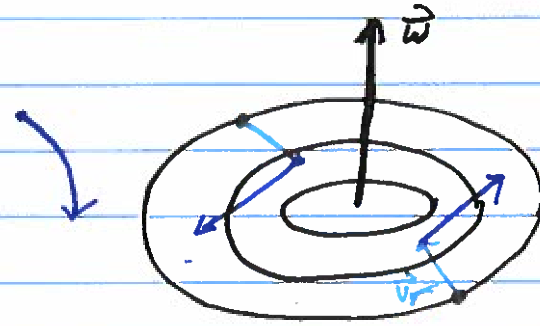
$$\boxed{\vec{F}_{eff} = m\vec{a}_r = \vec{F} - 2m(\vec{\omega}\times\vec{v}_r) - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

coriolis effect centrifugal effect



$-m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$

centrifugal effect — outward.



$-2m(\vec{\omega}\times\vec{v}_r)$

Coriolis effect

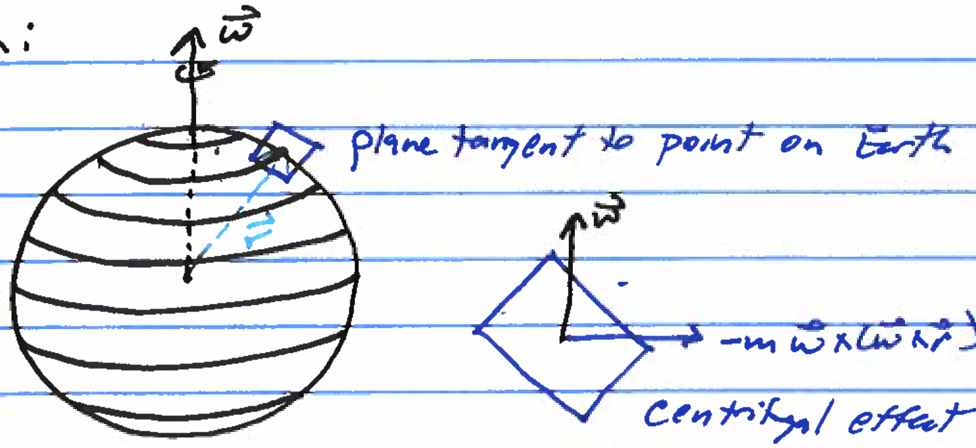
trajectory deflected to the right in rotating frame.

In the space frame, the tangential velocity (ωr) is faster further from the rotation axis.

From the perspective of the rotating frame, the object rotating with $\vec{\omega}$ has vanishing tangential velocity.

If the object now moves toward the rotation axis while maintaining its ωr tangential velocity (in the space frame), the object will appear to drift in the direction of the space-frame tangential velocity, according to the rotating observer.

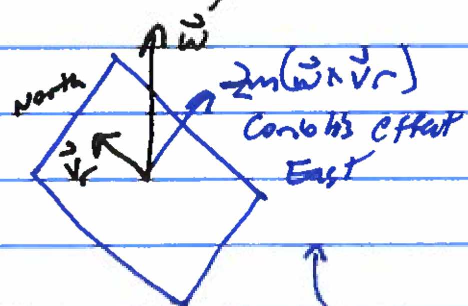
On Earth:



centrifugal effect
- perpendicular to latitude line, outward.

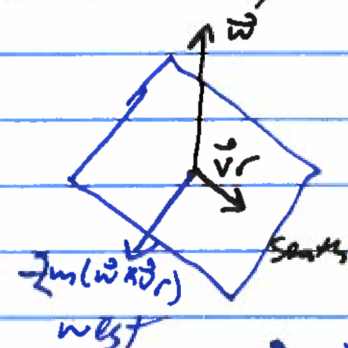
In northern hemisphere:

Air mass moving North:



Deflection East

Air mass moving South:



Deflection west

Hurricane Formation: Air moves toward Low-Pressure Center, deflected by Coriolis effect, generating circulation.

Northern Hemisphere:

Southern Hemisphere:

Let's consider some examples of the magnitude of these effects due to the rotational motion of Earth.

Earth rotates 2π radians in 24 hours relative to the radius vector to the sun. There are 365.25 such rotations in one year. This adds one complete rotation relative to the fixed stars in one year.

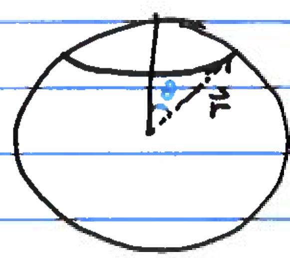
Relative to the fixed stars:

$$\omega = \left(\frac{2\pi}{24 \times 3600}\right)\left(\frac{366.25}{365.25}\right) s^{-1} = 7.292 \times 10^{-5}\ s^{-1}$$

At the equator, the centripetal acceleration is

$$\omega^2 r = 3.38\ cm/s^2 \quad \approx 0.003\, g$$

$\leftarrow 9.80\ m/s^2$



θ = co-latitude = $90° -$ latitude

At co-latitude θ ($=\measuredangle$ between $\vec{\omega}$ and $\vec{r}$),

$$|\vec{\omega} \times (\vec{\omega} \times \vec{r})| = \omega^2 r \sin\theta$$

$$= \omega^2 \times (\text{radius of circular motion})$$

Coriolis acceleration:

$$2|\vec{\omega}\times\vec{v}| < 2\omega v.$$

On Earth, with $\omega = 7.3\times10^{-5}\ s^{-1}$, for a velocity of magnitude 10^3 m/s, $2\omega v = 15\ cm/s^2 \approx 0.015 g$.

At the North pole, $a_{cor} = 2\omega v$

$\rightarrow$ linear deflection in time $t = \frac{1}{2}(2\omega v)t^2$

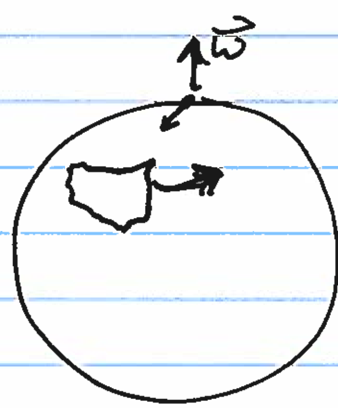
$$\text{Angular deflection} = \frac{\text{Linear deflection}}{\text{Distance travelled}} = \frac{\omega v t^2}{vt} = \omega t$$

This is the $\measuredangle$ of Earth's rotation in time t.

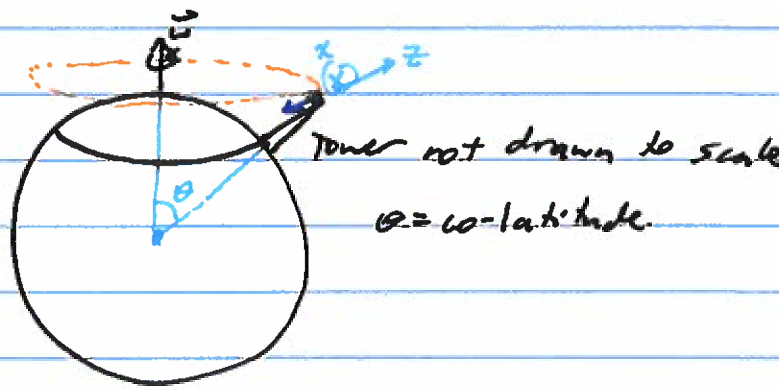
For a flight of 100s, deflection $\measuredangle = 7.3\times10^{-3}$ radians

$= 0.4°$

- Relevant for large projectiles/missiles/rockets.



Coriolis effect on vertically falling object:
Consider an object dropped from a tower in the Northern hemisphere:



Eastward speed at top of tower is larger than on ground. → Falling object dropped from tower appears deflected East in rotating (Earth) frame. (The same argument applies in the Southern hemisphere.)

Call x the East direction, z the upward vertical direction.

$$m\left(\frac{d^2x}{dt^2}\right)_r = -2m(\vec{\omega}\times\vec{v})_x$$
$$= -2m\,\omega\,v_z \sin\theta$$

With $v_z \approx -gt$, $\quad \frac{d^2x}{dt^2} = +2\omega g t \sin\theta$

z = distance fallen.

Integrate → $\quad x = \frac{\omega g}{3} t^3 \sin\theta \quad , \quad t = \sqrt{\frac{2z}{g}}$

$$x = \frac{\omega}{3}\sqrt{\frac{(2z)^3}{g}}\,\sin\theta.$$

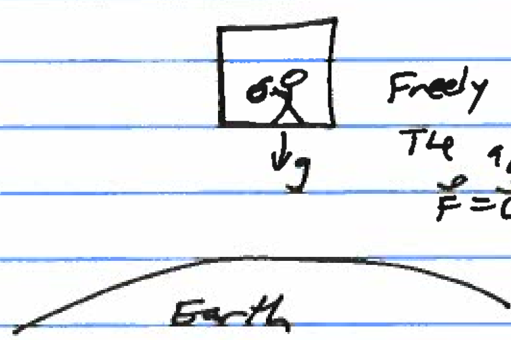
At the Equator, $\theta = \frac{\pi}{2}$, for $z = 100\,m$, $x = 2.2\,cm$.

Fictitious forces: Both the centrifugal and Coriolis effects are due to the transformation of the acceleration to a non-inertial reference frame.

The resulting fictitious forces arise from transformation of $m\vec{a}$, so they are proportional to the mass m on which they act.

The force of gravity is also proportional to m. Einstein observed that the effects of gravity can be undone locally by transforming to a freely falling frame.

In this sense, gravity is a fictitious force like the centrifugal and Coriolis forces, and the freely falling frames are the inertial frames.



Freely falling person with apple. The apple is stationary, with $\vec{F}=\vec{0}$, in the falling frame.