Infinitessimal Rotations

To analyze the dynamics of rigid body motion, we will look to describe how body axes change in time with respect to space axes.
Changes in orientation are described by rotation about some axis, so differential changes in orientation will be described by infinitessimal rotations.

The great simplification is that, whereas in general successive rotations do not commute, successive infinitessimal rotations do commute.

Consider a rotation $x_i' = a_{ij} x_j$

For an infinitessimal rotation, $a_{ij} = \delta_{ij} + \underbrace{\epsilon_{ij}}_{\text{small}}$.

We can write $\vec{x}' = (\mathbb{1} + \underline{\epsilon})\vec{x}$, where $\underline{\epsilon}$ is a matrix whose components are ϵ_{ij}.

For two successive rotations parametrized by $\underline{\epsilon}_1$, $\underline{\epsilon}_2$,

$$
\begin{aligned}
\vec{x}' &= (\mathbb{1} + \underline{\epsilon}_2)(\mathbb{1} + \underline{\epsilon}_1)\vec{x} \\
&= (\mathbb{1} + \underline{\epsilon}_2 + \underline{\epsilon}_1 + \underline{\epsilon}_2\underline{\epsilon}_1)\vec{x} \\
&\simeq (\mathbb{1} + \underline{\epsilon}_2 + \underline{\epsilon}_1)\vec{x} \\
&\simeq (\mathbb{1} + \underline{\epsilon}_1)(\mathbb{1} + \underline{\epsilon}_2)\vec{x}
\end{aligned}
$$

order reversed

with notation $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$,
the orthogonal transformation parametrized by Euler angles was

$$A = \begin{bmatrix} c_\psi c_\phi - c_\theta s_\phi s_\psi & c_\psi s_\phi + c_\theta c_\phi s_\psi & s_\psi s_\theta \\ -s_\psi c_\phi - c_\theta s_\phi c_\psi & -s_\psi s_\phi + c_\theta c_\phi c_\psi & c_\psi s_\theta \\ s_\theta s_\phi & -s_\theta c_\phi & c_\theta \end{bmatrix}$$

For infinitesimal $d\phi, d\theta, d\psi$:

$$A \approx \begin{bmatrix} 1 & (d\phi + d\psi) & 0 \\ -(d\phi + d\psi) & 1 & d\theta \\ 0 & -d\theta & 1 \end{bmatrix} \quad \text{antisymmetric}$$

If we write $A = \mathbb{1} + \underline{\epsilon}$, and define

$$\underline{\epsilon} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$$

then $\underline{\epsilon}_{ij} = \epsilon_{ijk}\, d\Omega_k$

Levi-Civita symbol

ϵ_{ijk} completely antisymmetric, $\epsilon_{123} = +1 = -\epsilon_{213}$

$\epsilon_{jik} = -\epsilon_{ijk}$, etc. $\epsilon_{112} = 0$, etc.

The vector $d\vec{\Omega} = \hat{i}\, d\theta + \hat{k}\,(d\phi + d\psi)$ by comparison with the matrix A above.

We can understand the antisymmetry of $\underline{\epsilon}$ from the orthogonality property of A:

$$A = 1 + \underline{\epsilon} \longrightarrow A^{-1} \approx \mathbb{1} - \underline{\epsilon}$$

Check: $AA^{-1} \approx (1+\underline{\epsilon})(1-\underline{\epsilon}) = 1 - \underline{\epsilon}\,\underline{\epsilon} \approx \mathbb{1}$. ✓

But for an orthogonal matrix $A^{-1} = A^T = \mathbb{1} + \underline{\epsilon}^T$.
Hence $\underline{\epsilon}^T = -\underline{\epsilon}$, i.e. $\underline{\epsilon}$ is antisymmetric.

Consider the change in a vector $\vec{r}$ under the infinitessimal transformation parametrized by $d\vec{\Omega}$.

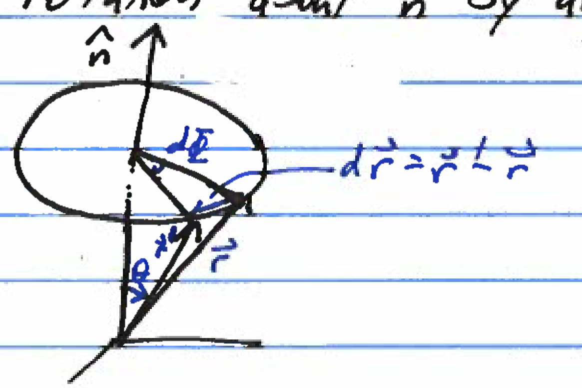
$$\vec{r}\,' - \vec{r} \equiv d\vec{r} = \underline{\epsilon}\,\vec{r}.$$

In components,
$$\begin{aligned} dx_i &= \underline{\epsilon}_{ij}\, x_j \\ &= \epsilon_{ijk}\, d\Omega_k\, x_j \\ &= (\vec{r} \times d\vec{\Omega})_i, \end{aligned}$$
where we used the general relation $(\vec{a} \times \vec{b})_i = \epsilon_{ijk}\, a_j b_k$.

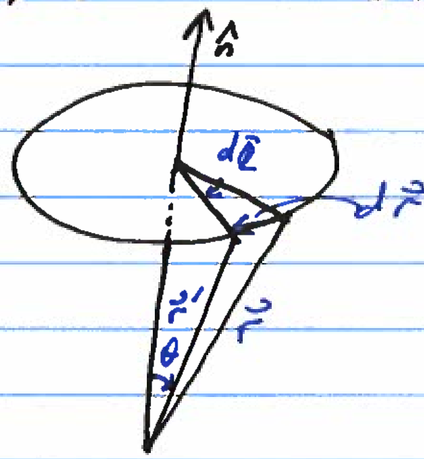
Hence, we can write $\boxed{d\vec{r} = \vec{r} \times d\vec{\Omega}.}$

Consider a rotation about $\hat{n}$ by angle $d\Phi$:

(Active clockwise rotation)



Geometrically, we see that $d\vec{r} = \vec{r} \times \hat{n}\, d\Phi$



Check:

$d\vec{r}$ is in direction $\perp$ plane spanned by $\vec{r}, \hat{n}$.

Magnitude: $|d\vec{r}| = (r \sin\theta)\, d\Phi = |\vec{r} \times \hat{n}\, d\Phi|$ ✓

Hence, we identify $\boxed{d\vec{\Omega} = \hat{n}\, d\phi.}$

active counterclockwise rotation by $d\Phi$

We can also write $\underline{\epsilon} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}(-d\Phi)$

$= n_i \underline{M}_i (-d\Phi)$,

where $\underline{M}_1 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\underline{M}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\underline{M}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ — Infinitesimal Rotation generators

Note that $\underline{M}_i \underline{M}_j - \underline{M}_j \underline{M}_i \equiv \boxed{[\underline{M}_i, \underline{M}_j] = \epsilon_{ijk} \underline{M}_k}$

– representation of Lie Algebra of rotation group

Rate of change of a vector:

We want to compare the rate of change of a vector $\vec{G}$ as seen by an observer attached to the body axes, with the rate of change of $\vec{G}$ as seen by an observer attached to the space frame.

The difference is a rotation of the body axes:

$$(d\vec{G})_{space} = (d\vec{G})_{body} + (d\vec{G})_{rot.}$$

For a vector fixed in the rigid body, $(d\vec{G})_{body} = \vec{0}$,
$(d\vec{G})_{space}$ is due entirely to the rotation of the body: (counterclockwise)

$$(d\vec{G})_{rot} = d\vec{\Omega} \times \vec{G}$$ ← sign from active counterclockwise rotation.

For an arbitrary vector, $(d\vec{G})_{space} = (d\vec{G})_{body} + d\vec{\Omega} \times \vec{G}$

The time rate of change of $\vec{G}$ as seen by the two observers is:

$$\left(\frac{d\vec{G}}{dt}\right)_{space} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \left(\frac{d\vec{\Omega}}{dt}\right) \times \vec{G}$$

$$\boxed{\frac{d\vec{G}}{dt} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \vec{\omega} \times \vec{G}}$$

$\vec{\omega}$ = instantaneous angular velocity, $\vec{\omega}\, dt = d\vec{\Omega}$.

To express $\vec{\omega}$ in the body frame:

Recall the Euler angles: ϕ (rotation about z)

θ (rot. about ξ')

ψ (rot. about z')

$\vec{\omega}_\phi$ in body frame is $A\begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}$

$$= \begin{pmatrix} \dot{\phi} \sin\theta \sin\psi \\ \dot{\phi} \sin\theta \cos\psi \\ \dot{\phi} \cos\theta \end{pmatrix}$$

$\vec{\omega}_\theta$ along line of nodes (in ξ'-direction)

$\rightarrow$ just apply final rotation about z'

$\Rightarrow$ $\vec{\omega}_\theta$ in body frame is $\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \dot{\theta} \cos\psi \\ -\dot{\theta} \sin\psi \\ 0 \end{pmatrix}$$

$\vec{\omega}_\psi$ along z' axis $\rightarrow$ $\vec{\omega}_\psi$ in body frame is

$$\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

Because infinitesimal rotations add,

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$$

$$= \begin{pmatrix} \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \dot{\phi} \cos\theta + \dot{\psi} \end{pmatrix}$$ in the body frame.