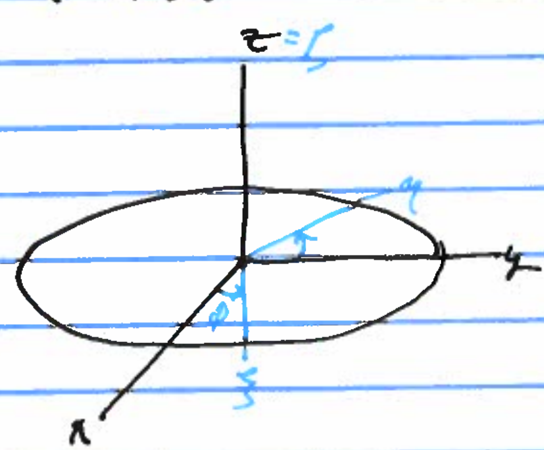
# The Euler Angles

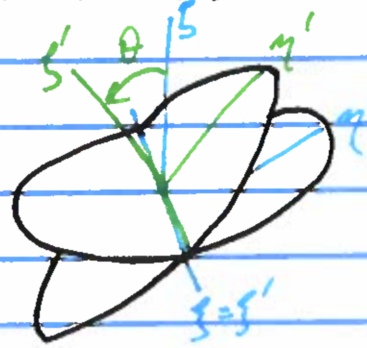
The 9 direction cosines provide an overcomplete description of the orientation of a rigid body. There are 6 relations between the direction cosines. In order to define a Lagrangian for the dynamics of a rigid body, we need a set of only 3 generalized coordinates describing the body orientation, in addition to the 3 coordinates describing its location, for example the position of the center of mass.

The Euler angles are one such set of generalized coordinates for the orientation of the body axes. They are described by considering three successive rotations from a fixed initial set of axes. There are various conventions.

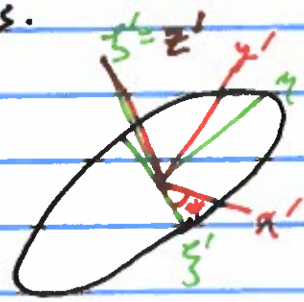
<u>x-convention</u>: 1) Rotate by angle $\phi$ (counterclockwise) about z-axis.



2) Call the rotated $x$-axis $\xi$. Rotate by $\theta$ about the $\xi$-axis.




3) Call the rotated $z$-axis $\zeta'$. Rotate by $\psi$ about the $\zeta'$ axis.

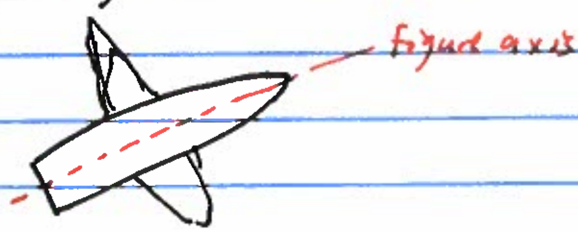



The angles $\phi, \theta, \psi$ parametrize the orientation of the coordinate axes $x', y', z'$. These angles are the Euler angles in the $x$-convention, so-called because the second rotation is about the rotated $x$-axis.

<u>$y$-convention</u>: The second rotation is about the rotated $y$-axis. Otherwise the sequence of rotations is the same.

Tait-Bryan angles: Used for describing orientation of aircraft, and other flying vehicles.

1) Rotate about vertical axis - gives heading or yaw angle.
2) Rotate about axis perpendicular to figure axis - gives pitch or attitude angle.
3) Rotate about the figure axis - gives roll or bank angle.

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We will work out the transformation matrix in the x-convention.

The net transformation is described by a product of the three transformation matrices for each of the three steps in the sequence.

Rotation by $\phi$ about $z$-axis: $\vec{\xi} = D\vec{x}$,

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation by $\theta$ about $\xi$-axis: $\vec{\xi}' = C\vec{\xi}$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Rotation by $\psi$ about $\xi'$: $\vec{x}' = B\vec{\xi}'$

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Net transformation: $\vec{x}' = A\vec{x}$, $\quad A = BCD$

Body frame (for $\vec{x}'$) ; space frame (for $\vec{x}$).

<u>Exercise:</u>

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

To transform back from body coordinates to space axes, use $\vec{x} = A^{-1}\vec{x}'$, with $A^{-1} = A^{T}$.

Other parametrizations are also possible.

In terms of Euler parameters (not angles) $e_0, e_1, e_2, e_3$, with $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$, the transformation matrix $A$ can be written,

$$A = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 + e_0 e_3) & 2(e_1 e_3 - e_0 e_2) \\ 2(e_1 e_2 - e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 + e_0 e_1) \\ 2(e_1 e_3 + e_0 e_2) & 2(e_2 e_3 - e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{bmatrix}$$

These are sometimes written in terms of the Cayley-Klein parameters $\alpha = e_0 + i e_3$, $\beta = e_2 + i e_1$.

## Euler's Theorem on Rigid Body Motion

Suppose $A(t)$ describes the orientation of body axes, with $A(0) = \mathbb{1}$, as a function of time $t$.

$$\boxed{\text{Euler's Theorem states that: The general displacement of a rigid body with one point fixed is a rotation about some axis.}}$$

In other words, $A$ represents a rotation.

Proof: Rotations have two defining characteristics: Any vector along the axis of rotation is unaffected; and the magnitude of vectors is unchanged.

Vector magnitudes are unchanged by orthogonal transformations:

Suppose $\vec{x}' = A\vec{x}$.

$$|\vec{x}'|^2 = |A\vec{x}|^2 = (A\vec{x})^T(A\vec{x})$$
$$= \vec{x}^T A^T A \vec{x}$$
$$= \vec{x}^T \mathbb{1} \vec{x}$$
$$= |\vec{x}|^2.$$

Aside: we wrote $(A\vec{x})^T = \vec{x}^T A^T$. We can understand this by writing components:

$$\underset{\substack{\uparrow \\ \text{row vector}}}{(A\vec{x})^T_i} \equiv \underset{\substack{\uparrow \\ \text{column vector}}}{(A\vec{x})_i} = A_{ij}\vec{x}_j = x_j A_{ij} = (x^T)_j (A^T)_{ji}$$
$$= (x^T A^T)_i$$

To prove Euler's theorem we need to show that $\exists$ a vector $\vec{R}'$ such that $\vec{R}' = A\vec{R} = \vec{R}$.

This is of the form of an eigenvalue equation, $A\vec{R} = \lambda\vec{R}$, with eigenvalue $\lambda = 1$.

Hence, Euler's theorem $\Leftrightarrow$ the orthogonal matrix $A$ has an eigenvalue $+1$.

Writing the eigenvalue equation as $(A - \lambda \mathbb{1})\vec{R} = 0$, this is a homogeneous equation for the components of $\vec{R}$.
$\vec{R} = \vec{0}$ is always a solution.
There are only nontrivial solutions for $\lambda$ satisfying

$$\boxed{\det(A - \lambda \mathbb{1}) = 0.}$$

This is the characteristic eqn, or secular eqn., for the matrix $A$. The solutions for $\lambda$ are the eigenvalues of $A$.

We will use orthogonality of $A$, namely $AA^T = A^TA = \mathbb{1}$, and the assumption that $A$ describes a proper transformation (because the body is rigid), so $\det A = +1. = \det A^T$.

Consider $(A - \mathbb{1})A^T = \mathbb{1} - A^T$

$$\det(A-\mathbb{1}) \underbrace{\det A^T}_{1} = \det(\mathbb{1} - A^T) = \det[-(A-\mathbb{1})]$$

$$\Rightarrow \det(A-\mathbb{1}) = \det[-(A-\mathbb{1})]$$

For $n \times n$ matrices, $\det(-M) = (-1)^n \det M$.
For the $3\times3$ matrix $A - \mathbb{1}$, $\det[(A-\mathbb{1})] = -\det(A-\mathbb{1})$

Hence we have $\det(A-\mathbb{1}) = -\det(A-\mathbb{1})$.
Therefore $\det(A-\mathbb{1}) = 0$.

$\Rightarrow \lambda = 1$ is an eigenvalue of $A$. ∎.

What about the other eigenvalues of $A$?
The matrix $A$ can be diagonalized by a similarity transformation.
Suppose $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are the eigenvectors of $A$, and $\lambda_1, \lambda_2, \lambda_3$ the corresponding eigenvalues.

We can think of $(\vec{x}_i)_j$ as the components of a matrix $X_{ij}$.

The eigenvalue equation is $\sum_j a_{ij} X_{jk} = \sum_j X_{ij} \delta_{jk} \lambda_k$

$$\equiv \sum_j X_{ij} \underline{\lambda}_{jk}$$

where the matrix $\underline{\lambda}$ is $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

In other words, $AX = X\underline{\lambda}$.

Multiply by $X^{-1}$: $\boxed{X^{-1} A X = \underline{\lambda}}$.

This is a similarity transformation.

$$\det(X^{-1} A X) = \det A = \det \underline{\lambda} = \lambda_1 \lambda_2 \lambda_3 .$$

One of the eigenvalues, say $\lambda_1 = 1$.
Hence, $\lambda_2 \lambda_3 = \det A = 1$. (proper orthogonal transformation)

$A$ is a real matrix $\rightarrow$ If $\lambda$ is an eigenvalue, so is $\lambda^*$.

$$\Rightarrow \lambda\lambda^* = 1 \rightarrow \lambda_2 = e^{i\Phi}, \quad \lambda_3 = e^{-i\Phi} \quad \text{for some } \Phi.$$

Axis of rotation: Set $\lambda = 1$, solve for $\vec{R}$ s.t. $A\vec{R} = \vec{R}$.

By a similarity transformation, transform $A$ s.t.

$$A' = S^{-1} A S = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A'(S^{-1}\vec{R}) = (S^{-1}AS)(S^{-1}\vec{R}) = S^{-1}\underbrace{(A\vec{R})}_{\vec{R}} = S^{-1}\vec{R} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathrm{Tr}\, A' = \mathrm{Tr}\, A = 1 + 2\cos\Phi = 1 + e^{i\Phi} + e^{-i\Phi}$$
$$= \lambda_1 + \lambda_2 + \lambda_3$$

The angle $\Phi$ in $A'$ is the phase angle of the eigenvalues (up to a sign).

Hence, we have determined the axis of rotation and the angle of rotation corresponding to the orthogonal transformation $A$.

> Generalization — <u>Chasles' Theorem</u>: The most general displacement of a rigid body is a translation plus a rotation.