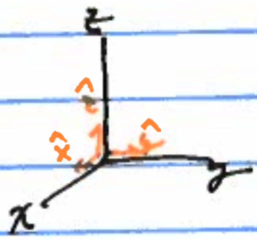
Review: Coordinate Systems, Holonomic vs. nonholonomic constraints, Rolling constraint, Conservation Laws, Effective potential

Coordinates – The most convenient choice of generalized coordinates is often dictated by the symmetries and constraints in a system.

Cartesian Coordinates:

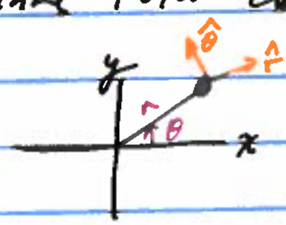


Kinetic Energy of particle:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Unit vectors: $\hat{x}, \hat{y}, \hat{z}$

Plane Polar Coordinates: Useful for rotational motion, rotational symmetry



$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$

Unit vectors:

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

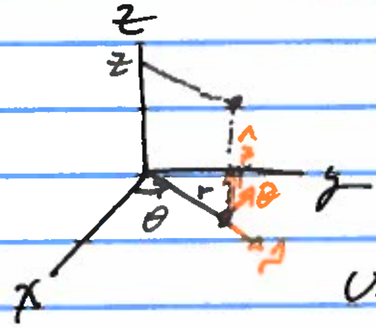
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\hat{x} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\hat{y} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

Kinetic Energy: $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

Cylindrical Coordinates: Useful for rotational motion when orthogonal direction is still relevant, or for systems w/ rotational symmetry about one axis.



$x = r\cos\theta$

$y = r\sin\theta$

z

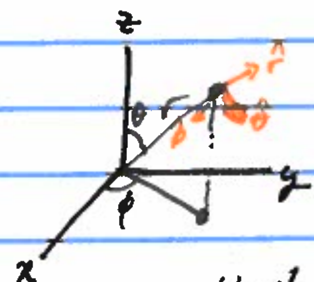
Unit vectors: $\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$

$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$\hat{z}$

Kinetic Energy: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

Spherical Coordinates: Useful for spherical symmetry, rotational motion in 3D.



$x = r\cos\phi\,\sin\theta$

$y = r\sin\phi\,\sin\theta$

$z = r\cos\theta$

Unit vectors: $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

Kinetic Energy: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

Hamilton's Principle: If $\{q_i\}$ is a set of independent (i.e. unconstrained) generalized coordinates, then in the absence of nonconservative forces, then the trajectory $q_i(t)$ subject to $q_i(t_1) = q_i^{(1)}$, $q_i(t_2) = q_i^{(2)}$ fixed satisfies the Euler-Lagrange equations that stationarize the action $I = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t)$, where

$$L(q_i, \dot{q}_i, t) = T - V(\{q_i\}, t) \quad \text{Lagrangian}$$

Example: $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \quad \text{Euler-Lagrange Eqs.}$$

$$\delta \vec{r} \Rightarrow m\frac{d}{dt}\dot{\vec{r}} + \nabla V = 0 \Rightarrow m\ddot{\vec{r}} = -\nabla V$$

Example: $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$$\delta r: \quad \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\delta \theta: \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

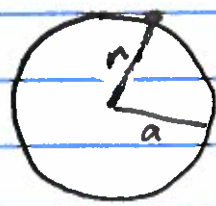
Constraints

Holonomic: Can be written in the form $f(\{q_i\}, t) = 0$

generalized coordinates

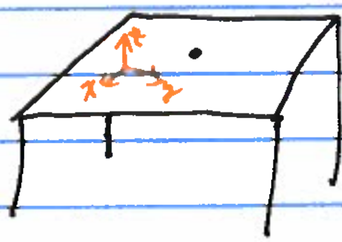
Examples:

Object on sphere of radius a



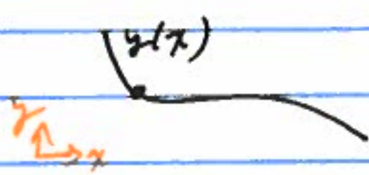
$$f(r, \theta, t) = r - a = 0$$

Object on a flat table



$$f(x, y, z, t) = z = 0$$

Bead on a 1-dimensional wire in a plane



$$f(x, y, t) = y - y(x) = 0$$

Non-holonomic constraints: Cannot be written as $f(\{q_i\}, t) = 0$

– Usually involve velocities.

Example: vertical disk rolling on a horizontal plane

← velocity of pt. on disk in contact w/ plane $= \vec{0}$.

$\Rightarrow$ velocity of center of disk has magnitude $v = a\dot{\phi}$

Lagrange Multipliers – for holonomic constraints

- Useful when 1) the solution to the constraint equation(s) is complicated;
  2) when you are interested in the forces of constraint.

Add to the Lagrangian $\sum_a \lambda_a f_a(\{q_i\}, t)$

Lagrange multiplier Constraint

$$I = \int_{t_1}^{t_2} dt\,\Big(L + \sum_a \lambda_a f_a(\{q_i\}, t)\Big) \quad \text{stationary}$$

Don't use constraint(s) to eliminate some q_i, until after deriving eqs. of motion!

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} - \sum_a \lambda_a \frac{\partial f_a}{\partial q_i} = 0$$

$\sum_a \lambda_a \frac{\partial f_a}{\partial q_i}$ = generalized force of constraint

For particle mechanics,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = \vec F \cdot \frac{\partial \vec r}{\partial q_i}, \text{ so } \sum_a \lambda_a \frac{\partial f_a}{\partial q_i} = \vec F \cdot \frac{\partial \vec r}{\partial q_i}$$

Force of constraint

Example: Bead on vertical loop of radius a, with gravity.

$$L = \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2) - mgr\sin\theta + \lambda(r-a)$$

Do not set $r=a$ here

Interpretation of λ: $\lambda \frac{\partial (r-a)}{\partial r} = \boxed{\lambda}$
$= \vec F \cdot \frac{\partial \vec r}{\partial r} \boxed{= F_r}$

λ = normal force enforcing the $r=a$ constraint.

Conservation Laws

Noether's Theorem (for systems w/o constraints):

If a transformation $q_i \to q_i + \epsilon \Delta q_i(\{q_j\}, t)$ leaves the action invariant up to $O(\epsilon)$, then the following holds:

for continuous ϵ.

$$\boxed{\frac{d}{dt}\underbrace{\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i\right)}_{\text{conserved quantity}} = 0}$$

If the Lagrangian is not invariant under such a transformation, but transforms by a total time derivative, i.e.

$$L \to L + \epsilon \frac{dF(\{q_i\}, t)}{dt},$$

then

$$\boxed{\frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\Delta q_i - F(\{q_i\}, t)\right) = 0}$$

Example: If $L(q_i, \dot{q}_i, t)$ is not explicitly dependent on t,

then under the transformation
$$q_i \to q_i(t) + \epsilon \dot{q}_i(t)$$
$$\dot{q}_i \to \dot{q}_i(t) + \epsilon \ddot{q}_i(t),$$

$$L \to L + \epsilon \frac{dL}{dt} + O(\epsilon^2)$$

The conserved quantity is the Hamiltonian:

$$\boxed{H = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \equiv \sum_i p_i \dot{q}_i - L}, \text{ where } \boxed{p_i \equiv \frac{\partial L}{\partial \dot{q}_i}}$$

A <u>cyclic coordinate</u> is a generalized coordinate that does not appear in the Lagrangian without time derivatives.

Suppose that generalized coordinate is called $q(t)$.

Then $q(t) \to q(t) + \epsilon$ is a symmetry (i.e. leaves the action invariant), and

$$\boxed{P_q = \frac{\partial L}{\partial \dot{q}}} \text{ is conserved.}$$

<u>Example</u>: particle in gravitational field of Earth

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

x, y — cyclic coordinates

$\to$ $P_x = m\dot{x}$, $P_y = m\dot{y}$ conserved

Central Forces:

potential $V(|\vec{r}|)$

- Depends on distance between masses, but not orientation.

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - V(|\vec{r}_2 - \vec{r}_1|)$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\vec{r}}^{\,2} - V(r)$$

Location of center of mass

— cyclic coordinate

$\mu \equiv \frac{m_1 m_2}{m_1+m_2}$ reduced mass

$\rightarrow \vec{P}_{\vec{P}_{cm}} = (m_1+m_2)\dot{\vec{R}}_{cm}$ conserved.

$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1+m_2}$$

Motion is in a plane $\perp$ $\vec{L}$ angular momentum

- conserved by rotational symmetry

— Motion about CM described by generalized coordinates (r, θ)

$$L = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}_{cm}^{\,2} + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

θ — cyclic coordinate

$\rightarrow \boxed{p_\theta = \mu r^2 \dot{\theta}}$ conserved — magnitude of $\measuredangle$ momentum

$\equiv \ell$

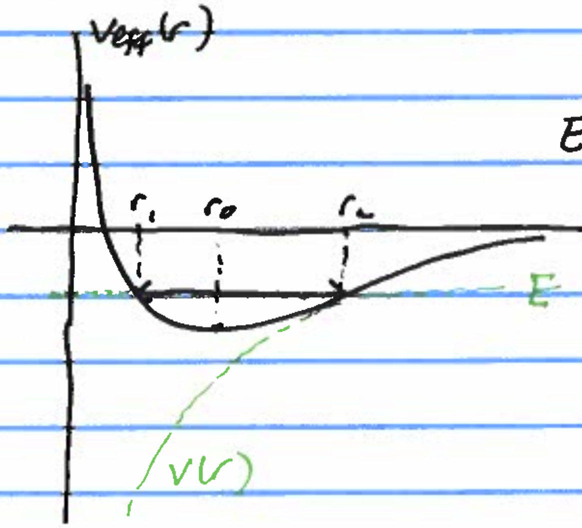
Lagrange Eqs:

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$= \mu\ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{\partial V}{\partial r} = \mu\ddot{r} + \frac{\partial}{\partial r}\underbrace{\left(\frac{\ell^2}{2\mu r^2} + V(r)\right)}$$

Effective potential $\quad V_{eff} = \frac{\ell^2}{2\mu r^2} + V(r)$

Example: $V(r) = -\frac{K}{r}$



$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

— conserved energy

$r = r_0$ — $V_{eff}'(r_0) = 0 \rightarrow \mu\ddot{r} = 0$

If $\dot{r} = 0$ initially $\rightarrow$ $r = r_0$ time-independent

— circular trajectory.

$E >$ min of V_{eff} $\rightarrow$ r varies with time between turning points r_1, r_2 where $E - V_{eff}(r) = 0$