PHYS 601 Classical Mechanics
F '21

Why study classical mechanics? As a grad student?

- It is a broad subject that is relevant for every subfield of physics.
- It is what more-precise descriptions of nature must reduce to in the appropriate regime.
- You are now well equipped to recognize the subtle assumptions made and to identify connections to other areas of physics.

Quantum: Hamiltonian ⟷ Schrödinger Eqn.
Lagrangian ⟷ Path Integral
Poisson Bracket ⟷ Commutation relations
Heisenberg Eqs. of motion
Hamilton-Jacobi Theory ⟷ de Broglie-Bohm interpretation

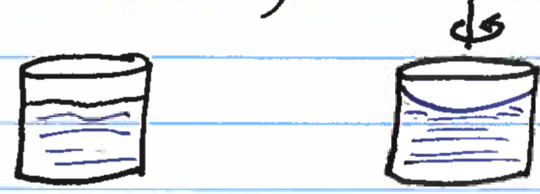
Relativity: Hamilton's Principle ⟷ Proper Time is maximized
→ Geodesic motion in spacetime.

Fluids
Chaos, nonlinear dynamics
Astronomy / Astrophysics
⋮
etc.

The things we assume in classical mechanics hide deep secrets of the universe.

<u>Example:</u> Inertial Reference Frames

- The rotating bucket of water.

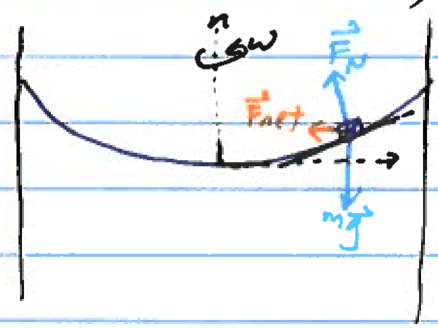


stationary rotating

Water in rotating bucket develops a concave surface.

Why? Model water as incompressible, no shear stresses

1) <u>Force argument</u>. Consider a bit of water at the surface a radial distance r from the central axis (the rotation axis), rotational speed ω.



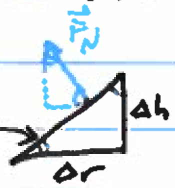
$\vec{F}_N$ = Normal force from water applied to the surface.

$m\vec{g}$ = weight of bit of water

$\vec{F}_{net}$ = centripetal force

Radial direction: $F_{Nr} = F_{net} = -m\omega^2 r$

Vertical direction: $F_{Nz} = mg$



tangent to surface

<u>Normal force:</u> $\left|\frac{F_{Nr}}{F_{Nz}}\right| = \frac{\omega^2 r}{g} = \frac{dh}{dr}$

$$\Rightarrow \boxed{h(r) = \frac{\omega^2}{2g} r^2 + \text{const.}}$$

2) Equipotential Argument. Consider the rotating frame.

The bit of water is stationary in the rotating frame, but there is a fictitious centrifugal force.

$$\vec{F}_c = +m\omega^2 r \ \hat{r}$$

conservative: $\vec{F}_c = -\nabla V_c(r)$, $V_c(r) = -\frac{1}{2} m\omega^2 r^2$

Potential energy:

$$V(r) = V_0 + mgh(r) - \frac{1}{2} m\omega^2 r^2$$

Equilibrium: water surface is an equipotential

$$V(r) = \text{const.}$$

$$\rightarrow \boxed{h(r) = \frac{\omega^2}{2g} r^2 + \text{const.}}, \text{ as before.}$$

Question: The bucket is rotating, but relative to what?

Newton — The rotation is relative to a fixed, absolute space. In contrast, inertial frames* are at rest or in uniform motion relative to absolute space.

Mach — The average distribution of matter in the universe, or the fixed stars, determine the inertial frames.

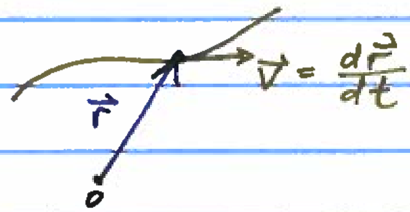
The stars appear to rotate in the rotating frame.

Einstein — The local distribution of matter determines local inertial frames. Small freely falling objects are inertial

*The term "reference frame" is due to James Thomson, 1884

Elementary Principles (Goldstein, Ch. 1)

Particle mechanics:



Linear momentum $\vec{p} = m\vec{v}$

Newton's Second Law of Motion $\vec{F} = \frac{d\vec{p}}{dt} \equiv \dot{\vec{p}}$

Note: In order for Newton's Second Law to be more than a definition of $\vec{F}$, we need to be able to model forces that act equivalently on different particles, e.g. spring $F = -kx$.

Note: If particle mass is constant then $\vec{F} = m\frac{d\vec{v}}{dt} \equiv m\vec{a}$.

Conservation of momentum: If total force $\vec{F} = \vec{0}$, then
$\dot{\vec{p}} = \vec{0}$, $\vec{p} = \text{constant}$.

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$ — depends on origin O

Torque about O:
$$\begin{aligned}\vec{N} &= \vec{r} \times \vec{F} \\ &= \vec{r} \times \frac{d}{dt}(m\vec{v}) = \cancel{\vec{v} \times m\vec{v}} + \vec{r} \times \frac{d}{dt}(m\vec{v}) \\ &= \frac{d}{dt}(\vec{r} \times m\vec{v}) \\ &= \frac{d\vec{L}}{dt} \equiv \dot{\vec{L}}\end{aligned}$$

<u>Conservation of angular momentum</u>: If torque $\vec{N}=0$ then $\dot{\vec{L}}=0$, angular momentum is conserved.

<u>Work</u> done by external force $\vec{F}$ as particle moves from point 1 to point 2:

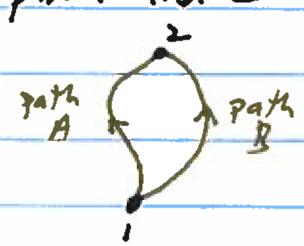
$$W_{12} \equiv \int_1^2 \vec{F}\cdot d\vec{s}$$

constant particle mass $\rightarrow$ $W_{12} = \int \left(m\frac{d\vec{v}}{dt}\right)\cdot(\vec{v}\,dt)$

$$= \frac{m}{2}\int \frac{d}{dt}(\vec{v}\cdot\vec{v})\,dt$$
$$= \frac{m}{2}(v_2^2 - v_1^2)$$
$$\equiv T_2 - T_1$$

<u>Kinetic energy</u> $T \equiv \frac{m}{2}v^2$

<u>Conservative Force</u>: Any force for which work done bet. pts. 1 and 2 is independent of path.



$W_{12}(\text{path } A) = W_{12}(\text{path } B)$ for conservative force.

<u>Note</u>: Under any closed path (i.e. pt. 1 = pt. 2),
$W = \oint \vec{F}\cdot d\vec{s} = 0$. (for conservative force)

$\rightarrow$ $\vec{F} = -\nabla V(\vec{r})$ for some potential energy function $V(\vec{r})$

note sign convention.

For conservative force $\vec{F} = -\nabla V$, work $W_{12} = \int -\nabla V \cdot d\vec{s}$

$$= -\int \frac{\partial V}{\partial s}\, ds$$

$$\left.\begin{aligned} W_{12} &= V_1 - V_2 \\ \text{But recall } W_{12} &= T_2 - T_1 \end{aligned}\right\} T_1 + V_1 = T_2 + V_2$$

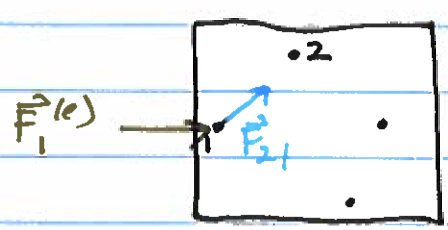
$\Rightarrow$ <u>Conservation of Energy</u>: If only conservative forces act on a particle, then the mechanical energy $T+V$ is conserved.

Next consider a <u>system of particles</u>.

Newton's 2nd Law: $\dot{\vec{p}}_i = \sum_j \vec{F}_{ji} + \vec{F}_i^{(e)}$

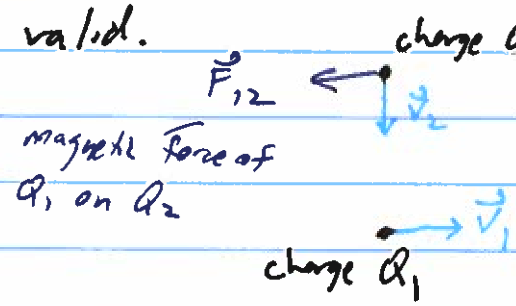
Internal force due to particle j on particle i

External force on particle i



<u>Newton's 3rd Law:</u> $\vec{F}_{ji} = -\vec{F}_{ij}$

Note that this form of Newton's 3rd Law is not always valid.



$\vec{F}_{21} = \vec{0}$.

Suppose $\vec{F}_{ii} = \vec{0}$, $\vec{F}_{ij} = -\vec{F}_{ji}$.

$\dot{\vec{p}}_i = \sum_j \vec{F}_{ji} + \vec{F}_i^{(e)}$. Sum on particles i:

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \cancel{\sum_{i \neq j} \vec{F}_{ji}} + \sum_i \vec{F}_i^{(e)}$$

Define $\vec{R} \equiv \dfrac{\sum m_i \vec{r}_i}{\sum m_j} = \dfrac{\sum m_i \vec{r}_i}{M}$

where $M = \sum_i m_i$ total mass

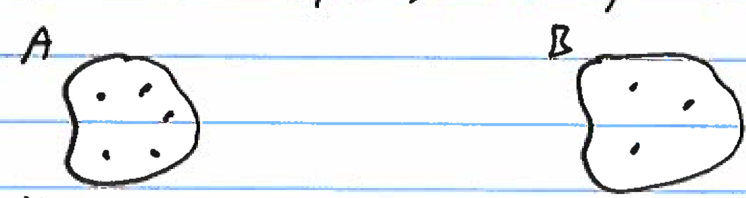
$\vec{R}$ = center of mass

$$\boxed{M \frac{d^2 \vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)}}$$ net external force

- Resembles Newton's 2nd Law for single particle.

We should emphasize this remarkable point.

Consider two systems of particles



$M_A = \sum_{i \in \text{system } A} m_i$ $\qquad$ $M_B = \sum_{j \in \text{system } B} m_j$

$\vec{R}_A$ = CM of system A $\qquad$ $\vec{R}_B$ = CM of system B

$$M_A \frac{d^2 \vec{R}_A}{dt^2} = \underbrace{\vec{F}_{BA}}_{\sum_{i \in A,\, j \in B} F_{ji}} + \underbrace{\vec{F}_A^{(e)}}_{\text{External force, not from system B.}}$$

Newton's Laws reproduce themselves for larger systems.

Total linear momentum $\vec{P} = \sum_i m_i \frac{d\vec{r}_i}{dt} = M \frac{d\vec{R}}{dt}$

Conservation of total linear momentum: If the total external force vanishes, then $\vec{P}$ is conserved.

Total angular momentum: $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$

$$\dot{\vec{L}} = \sum \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum \vec{r}_i \times \dot{\vec{p}}_i$$

(because $\dot{\vec{r}}_i \times \vec{p}_i = \vec{0}$.)

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{\text{pairs } \langle ij \rangle} (\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij})$$

$$= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{\langle ij \rangle} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$$

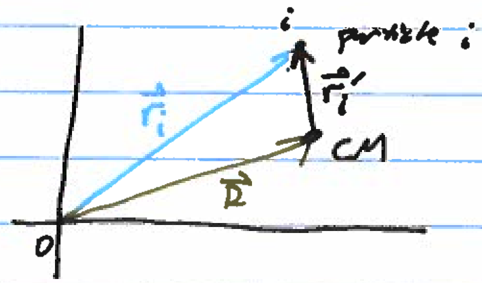
Assume $\vec{F}_{ji}$ is in the direction $\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$ (or opposite that direction).

Then $(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = \vec{0}$. (Note that this is not always valid. Consider magnetic forces.)

Then $\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} \equiv \vec{N}^{(e)}$ net torque

Conservation of Total Angular Momentum: If $\vec{N}^{(e)} = \vec{0}$, then $\vec{L}$ is conserved.

Decomposition in terms of motion of center of mass and motion about the center of mass:



$\vec r_i' = \vec r_i - \vec R$ = displacement of particle i from CM

$$\vec v_i = \vec v_i' + \vec V$$

$\vec V = \frac{d\vec R}{dt}$ velocity of CM

$\vec v_i' = \frac{d\vec r_i'}{dt}$ velocity relative to CM.

$$\vec L = \sum_i \vec R \times m_i \vec V + \sum_i \vec r_i' \times m_i \vec v_i' + \cancel{\sum_i m_i \vec r_i' \times \vec V} + \cancel{\vec R \times \frac{d}{dt}\sum_i m_i \vec r_i'}$$

Note $\sum m_i \vec r_i' = \vec 0$.
- CM in frame where CM is at the origin

$$\Rightarrow \boxed{\vec L = M\vec R \times \vec V + \sum \vec r_i' \times \vec p_i'}$$

Kinetic Energy: $T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i (\vec V + \vec v_i')\cdot(\vec V + \vec v_i')$

$$= \frac{1}{2}\sum_i m_i V^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \cancel{\sum_i \vec V \cdot \frac{d}{dt}(m_i \vec r_i')}$$

$$T = \underbrace{\tfrac{1}{2}MV^2}_{\text{center of mass motion}} + \underbrace{\tfrac{1}{2}\sum_i m_i v_i'^2}_{\text{motion about center of mass.}}$$

If $\vec{F}_i^{(e)}$ is conservative, then $\vec{F}_i^{(e)} = -\nabla_i V_i$ for some V_i

derivatives w.r.t. $\vec{r}_i$

If $\vec{F}_{ij} = -\vec{F}_{ji}$ is conservative and in the direction $\vec{r}_i - \vec{r}_j$ (or opposite that direction), then

$\vec{F}_{ji} = -\nabla_i V_{ij}$, with $V_{ij} = V_{ij}(|\vec{r}_i - \vec{r}_j|)$

Exercise: Define $V = \sum_i V_i + \sum_{\text{pairs } \langle ij \rangle} V_{ij}$,

show that the total mechanical energy $T+V$ is conserved.

Note: For a rigid body, $\sum_{\langle ij \rangle} V_{ij} = \text{constant}$,

$|\vec{r}_i - \vec{r}_j| = \text{constant}$.

Internal forces do no work in a rigid body.