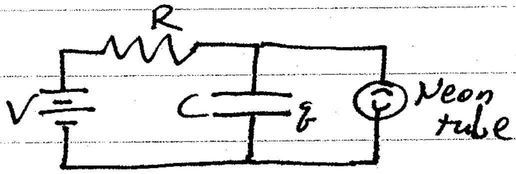
Phys 475 S'10 Problem Set 9 Solutions

7.13.4



a) Check that $q(t) = CV(1-e^{-t/RC})$ satisfies

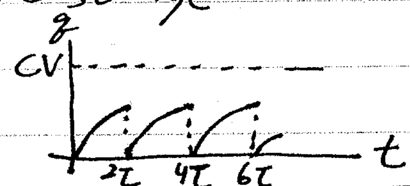
$$R\frac{dq}{dt} + q/C = V.$$

$$R\frac{dq}{dt} + q/C = \cancel{R}\cdot\cancel{C}V\cdot\frac{1}{\cancel{RC}}e^{-t/RC} + \frac{1}{\cancel{C}}\cancel{C}V(1-e^{-t/RC})$$
$$= V \quad \checkmark$$

At $t=0$, $q = CV(1-e^{-0/RC}) = 0$ ✓

Hence, $q = CV(1-e^{-t/RC})$ describes the charge on the capacitor while charging (i.e. before discharge)

b) Discharge occurs at $t = \boxed{\frac{1}{2}RC = 2\tau}$ (period)



c) $\boxed{q(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/\tau}}$, $c_n = \frac{1}{2\tau}\int_0^{2\tau} q(t)e^{-in\pi t/\tau}dt$

$$c_n = \frac{1}{2\tau}\int_0^{2\tau} CV(1-e^{-t/4\tau})e^{-in\pi t/\tau}\,dt$$
$$= \frac{CV}{2\tau}\left[\underbrace{\frac{\tau}{-in\pi}e^{-in\pi t/\tau}\Big|_0^{2\tau}} + \frac{\tau}{1/4 + in\pi}e^{-t/4\tau - in\pi t/\tau}\Big|_0^{2\tau}\right]$$
$$= \frac{CV}{2\tau}\left[2\tau\,\delta_{n0} + \frac{4\tau}{1+4in\pi}(e^{-1/2}-1)\right]$$

(using $e^{-2n\pi i} = 1$)

$$\boxed{c_n = \frac{2CV}{1+16n^2\pi^2}(1-4in\pi)(e^{-1/2}-1)}$$

12.1.4 $y'' = -4y$. Assuming $y = Ae^{ikx}$, $y'' = -Ak^2 e^{ikx}$

$-k^2 y = -4y \rightarrow k = \pm 2$

$y = Ae^{2ix} + Be^{-2ix}$, or $\boxed{y = \tilde{A}\sin 2x + \tilde{B}\cos 2x}$

Series solution: Assume $y = \sum_{n=0}^{\infty} a_n x^n$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$y'' = -4y \rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = -\sum_{n=0}^{\infty} 4 a_n x^n$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n = -\sum_{n=0}^{\infty} 4 a_n x^n$$

$$\rightarrow a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}$$

n even: $a_2 = \frac{-4a_0}{2\cdot 1}$, $a_4 = \frac{-4a_2}{4\cdot 3} = \frac{(-4)(-4)a_0}{4\cdot 3\cdot 2\cdot 1}$

$$a_{2n} = \frac{(-4)^n}{(2n)!} a_0 = (-1)^n \frac{2^{2n}}{(2n)!} a_0$$

n odd: $a_3 = \frac{-4a_1}{3\cdot 2}$, $a_5 = \frac{-4a_3}{5\cdot 4} = \frac{(-4)(-4)}{5\cdot 4\cdot 3\cdot 2} a_1$

$$a_{2n+1} = \frac{(-1)^n 2^{2n} a_1}{(2n+1)!} = \frac{(-1)^n 2^{2n+1}}{2(2n+1)!} a_1$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} a_0 + \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \frac{a_1}{2}$$

$$\boxed{y = a_0 \cos 2x + \frac{a_1}{2} \sin 2x}$$

$\rightarrow \tilde{A} = a_0$, $\tilde{B} = a_1/2$

12.4.1 $P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

Let $v = (x^2-1)^\ell$

$$(x^2-1)\frac{dv}{dx} = (x^2-1)\ell(x^2-1)^{\ell-1}\cdot 2x = 2\ell x v$$

$$\frac{d^{\ell+1}}{dx^{\ell+1}}\left[(x^2-1)\frac{dv}{dx}\right] = 2\ell \frac{d^{\ell+1}}{dx^{\ell+1}}\left[x v\right]$$

$$(x^2-1)\frac{d^{\ell+2}v}{dx^{\ell+2}} + \binom{\ell+1}{1}\frac{d}{dx}(x^2-1)\frac{d^\ell v}{dx^\ell}$$
$$+\binom{\ell+1}{2}\frac{d^2}{dx^2}(x^2-1)\frac{d^{\ell-1}v}{dx^{\ell-1}}$$
$$= 2\ell x \frac{d^{\ell+1}v}{dx^{\ell+1}} + 2\ell\binom{\ell+1}{1}\frac{dx}{dx}\frac{d^\ell v}{dx^\ell}$$

$$(x^2-1)\frac{d^{\ell+2}v}{dx^{\ell+2}} + (\ell+1)(2x)\frac{d^\ell v}{dx^\ell} + \frac{(\ell+1)(\ell)}{2!}\cdot 2\frac{d^{\ell-1}v}{dx^{\ell-1}}$$
$$= 2\ell x \frac{d^{\ell+1}v}{dx^{\ell+1}} + 2\ell\frac{d^\ell v}{dx^\ell}$$

Subtracting the left-hand side from both sides gives the desired result:

$$\boxed{(1-x^2)\left(\frac{d^\ell v}{dx^\ell}\right)'' - 2x\left(\frac{d^\ell v}{dx^\ell}\right)' + \ell(\ell+1)\frac{d^\ell v}{dx^\ell} = 0}$$

Hence, $\frac{d^\ell v}{dx^\ell}$ satisfies Legendre's equation.

12.5.9 $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$\left.\begin{aligned} 3x^2 &= 2P_2(x) + 1 \\ x &= P_1(x) \\ -1 &= -P_0(x) \end{aligned}\right\} \quad 3x^2 + x - 1 = 2P_2(x) + P_1(x) + 1 - P_0(x) = \boxed{2P_2(x) + P_1(x)}$$

12.6.5 $\int_{-1}^{1} P_0(x) P_2(x)\,dx = \int_{-1}^{1} 1 \cdot \frac{1}{2}(3x^2 - 1)\,dx$

$$= \frac{1}{2}\left(\frac{3x^3}{3} - x\right)\Big|_{-1}^{1} = \frac{1}{2}\left((1^3 - 1) - ((-1)^3 - (-1))\right)$$

$$= 0 \quad \checkmark$$

12.7.5 If $\ell > 0$ then $\int_{-1}^{1} P_\ell(x)\,dx = \int_{-1}^{1} P_\ell(x) P_0(x)\,dx = 0$

by orthogonality.

12.9.14 We want the Legendre series for $|x|$ to second order in x:

$$|x| = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x), \quad c_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} |x| P_\ell(x)\,dx$$

$$c_0 = \frac{1}{2}\int_{-1}^{1} |x| \cdot 1\,dx = \frac{1}{2} \cdot 2 \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} |x|\, x\,dx = \frac{3}{2}\left[\int_{-1}^{0} -x^2\,dx + \int_0^1 x^2\,dx\right] = 0$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} |x| \cdot \frac{1}{2}(3x^2 - 1)\,dx = \frac{5}{2} \cdot 2 \int_0^1 \frac{1}{2} x(3x^2 - 1)\,dx$$

$$= \frac{5}{2}\left(\frac{3x^4}{4} - \frac{x^2}{2}\right)\Big|_0^1 = \frac{5}{2}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{5}{8}$$

Best fit: $|x| = \frac{1}{2}\underset{P_0(x)}{1} + \frac{5}{8} \cdot \underbrace{\frac{1}{2}(3x^2 - 1)}_{P_2(x)} = \boxed{\frac{3}{16} + \frac{15}{16}x^2}$