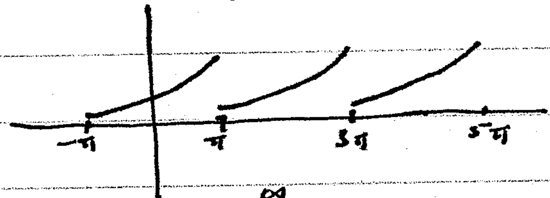
Phys 975 S'10 Problem Set 8 Solutions

7.8.12 a) $f(x) = e^{x}, \quad -\pi < x < \pi$



$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}^{\infty} a_n \cos(nx) + \sum_{n\geq 1}^{\infty} b_n \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \cos(nx)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \cdot \frac{e^{inx} + e^{-inx}}{2}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{x(1+in)} + e^{x(1-in)}\right) dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+in} e^{x(1+in)}\Big|_{-\pi}^{\pi} + \frac{1}{1-in} e^{x(1-in)}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+in}\left(e^{\pi} e^{in\pi} - e^{-\pi} e^{-in\pi}\right) + \frac{1}{1-in}\left(e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n}{1+n^2}(1-in)\left(e^{\pi} - e^{-\pi}\right) + \frac{(-1)^n}{1+n^2}(1+in)\left(e^{\pi} - e^{-\pi}\right)\right]$$

$$= \frac{1}{2\pi}\,\frac{(-1)^n \cdot 2\left(e^{\pi} - e^{-\pi}\right)}{1+n^2} = \boxed{\frac{(-1)^n}{\pi(1+n^2)} \cdot 2\sinh\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\sin(nx)\,dx = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\left(e^{x(1+in)} - e^{x(1-in)}\right)dx$$

$$= \frac{1}{2\pi i}\left[\frac{(-1)^n}{1+n^2}(1-in)\left(e^{\pi} - e^{-\pi}\right) - \frac{(-1)^n}{1+n^2}(1+in)\left(e^{\pi} - e^{-\pi}\right)\right]$$

$$= \frac{(-1)^{n+1}}{2\pi i}\,\frac{1}{(1+n^2)}(2in)\left(e^{\pi} - e^{-\pi}\right)$$

$$= \boxed{\frac{(-1)^{n+1}}{\pi(1+n^2)} 2n\sinh\pi}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-in)}\, dx$$

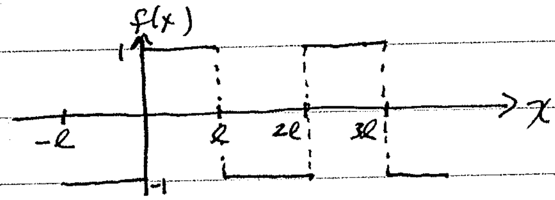
$$= \frac{1}{2\pi} \frac{1}{1-in} e^{x(1-in)} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \frac{(-1)^n}{1+n^2} (1+in) \left(e^{\pi} - e^{-\pi} \right)$$

$$= \frac{(-1)^n}{\pi(1+n^2)} (1+in) \sinh \pi$$

7.9.6

$$f(x) = \begin{cases} -1, & -\ell < x < 0 \\ 1, & 0 < x < \ell \end{cases}$$



f(x) is odd about $x = 0, \ell$ w/ period 2ℓ.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}, \quad b_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} 1 \cdot \sin \frac{n\pi x}{\ell}\, dx$$

$$= \frac{2}{\ell}\left(-\frac{\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right)\Bigg|_0^{\ell}$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - 1\right) = -\frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$f(x) = \frac{4}{\pi}\sin\frac{\pi x}{\ell} + \frac{4}{3\pi}\sin\frac{3\pi x}{\ell} + \frac{4}{5\pi}\sin\frac{5\pi x}{\ell} + \cdots$$

$$= \sum_{n=0}^{\infty} C_n \sin\frac{(2n+1)\pi x}{\ell}, \text{ with}$$

$$\boxed{C_n = \frac{4}{(2n+1)\pi}}$$

C_n in problem 7.11.5

7.11.5 $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$

$= \frac{\pi^2}{16} \sum_{n=0}^{\infty} c_n^2$, with c_n from problem 7.9.6

Parseval's Thm: $\frac{1}{2} \sum_n c_n^2 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)^2 \, dx$

$= \frac{1}{2\ell} \int_{-\ell}^{\ell} 1 \cdot dx = 1$

$\Rightarrow \sum_n c_n^2 = 2$

$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{16} \cdot 2 = \boxed{\frac{\pi^2}{8}}$

7.12.34 $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \, g(\alpha) e^{i\alpha x}$, $\overline{f(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \, \overline{g(\alpha)} e^{-i\alpha x}$

$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, f(x) e^{-i\alpha x}$

$\boxed{\int_{-\infty}^{\infty} dx \, |f(x)|^2} = \int_{-\infty}^{\infty} dx \, \overline{f(x)} f(x)$

$= \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \, \overline{g(\alpha)} e^{-i\alpha x} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha' \, g(\alpha') e^{i\alpha' x}$

$= \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\alpha' \frac{1}{2\pi} \overline{g(\alpha)} g(\alpha') \underbrace{\int_{-\infty}^{\infty} dx \, e^{-ix(\alpha - \alpha')}}_{2\pi \delta(\alpha - \alpha')}$

$= \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\alpha' \, \overline{g(\alpha)} g(\alpha') \, \delta(\alpha - \alpha')$

$= \int_{-\infty}^{\infty} d\alpha \, \overline{g(\alpha)} g(\alpha) \boxed{= \int_{-\infty}^{\infty} d\alpha \, |g(\alpha)|^2}$

7.12.35 $\alpha = \frac{2\pi p}{h}$, $f(x) = \psi(x)$, $g(\alpha) = \sqrt{\frac{h}{2\pi}}\,\phi(p)$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha\, g(\alpha)\, e^{i\alpha x}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \cdot \frac{2\pi}{h} \sqrt{\frac{h}{2\pi}}\,\phi(p)\, e^{i\frac{2\pi p}{h}x}$$

$$= \boxed{\frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} dp\, \phi(p)\, e^{2\pi i p x/h}}$$

$$\phi(p) = \sqrt{\frac{2\pi}{h}}\, g(\alpha) = \sqrt{\frac{2\pi}{h}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-i\alpha x}$$

$$= \boxed{\frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} dx\, \psi(x)\, e^{-2\pi i p x/h}}$$

Parseval's theorem: $\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} d\alpha\, |g(\alpha)|^2$

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = \int_{-\infty}^{\infty} dp\, \frac{2\pi}{h} \left| \sqrt{\frac{h}{2\pi}}\,\phi(p) \right|^2$$

$$= \boxed{\int_{-\infty}^{\infty} dp\, |\phi(p)|^2}$$

7.13.18 $f(x) = \int_{-\infty}^{\infty} d\alpha\, g(\alpha)\, e^{i\alpha x}$

$$\frac{df}{dx} = \int_{-\infty}^{\infty} d\alpha\, g(\alpha) \frac{d}{dx} e^{i\alpha x} = \int_{-\infty}^{\infty} d\alpha\, \underbrace{g(\alpha)\, i\alpha}_{\text{Fourier transform of } \frac{df}{dx}}\, e^{i\alpha x}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(x) \frac{df}{dx} dx = \int_{-\infty}^{\infty} d\alpha\, \bar{g}(\alpha) \cdot i\alpha g(\alpha)$$ (by generalized Parseval)

$$= \int_{-\infty}^{\infty} d\alpha\, i\alpha\, |g(\alpha)|^2$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} d\alpha\, \alpha\, |g(\alpha)|^2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dx\, \bar{f}(x) \frac{df}{dx}}$$