Phys 475 S'10 Problem Set 7 Solutions

6.7.19 Let $\vec{V} = \vec{r}/|\vec{r}|$. $\vec{V}$ is a unit vector in the radial direction, i.e. $\vec{V} = \hat{e}_r$. (in spherical coordinates)

In components, $V_r = 1$, $V_\theta = V_\phi = 0$.

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot 1) = \boxed{2/r}, \text{ where } r = |\vec{r}|$$

Alternatively, using Cartesian coordinates,

$$\vec{V} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})$$

$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left[\left(1 - \frac{x^2}{x^2+y^2+z^2}\right) + \left(1 - \frac{y^2}{x^2+y^2+z^2}\right) + \left(1 - \frac{z^2}{x^2+y^2+z^2}\right)\right]$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}} = \boxed{\frac{2}{r}}$$

6.10.11 $\boxed{\int_{\partial V} \vec{B} \cdot \hat{n}\, d\sigma} = \int_{\partial V} (\nabla \times \vec{A}) \cdot \hat{n}\, d\sigma$ where ∂V = boundary of volume V.

$= \int_V \nabla \cdot (\nabla \times \vec{A})\, dV$ by the divergence theorem.

$\boxed{= 0}$ because $\nabla \cdot (\nabla \times \vec{A}) = 0$ for any $\vec{A}$.

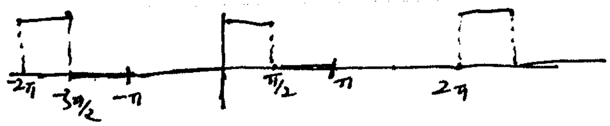
6.10.16 $\boxed{\int_V (\phi \nabla^2 \psi + \nabla\phi \cdot \nabla\psi)\, dV} = \int_V \nabla \cdot (\phi \nabla \psi)\, dV$

$\boxed{= \int_{\partial V} (\phi \nabla \psi) \cdot \hat{n}\, d\sigma}$, where ∂V is the closed surface bounding V.

$\boxed{\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV} = \int_V \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi)\, dV$

$$\boxed{= \int_{\partial V} (\phi \nabla \psi - \psi \nabla \phi) \cdot \hat{n}\, d\sigma}$$

2.5.2 $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$



$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(nx) + \sum_{n\geq 1} b_n \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \cos(nx)\,dx = \frac{1}{\pi n}\sin(nx)\Big|_0^{\pi/2}$$

$$= \frac{1}{\pi n}\sin\frac{n\pi}{2}, \quad n > 0 \qquad \left(a_0 = \frac{1}{2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \sin(nx)\,dx = -\frac{1}{\pi n}\cos(nx)\Big|_0^{\pi/2}$$

$$= \frac{1}{\pi n}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$\boxed{f(x) = \frac{1}{4} + \sum_{n\geq 1} \frac{1}{\pi n}\sin\frac{n\pi}{2}\cos nx + \sum_{n\geq 1}\frac{1}{\pi n}\left(1 - \cos\frac{n\pi}{2}\right)\sin nx}$$

$$= \frac{1}{4} + \frac{1}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x + \dots\right)$$

$$+ \frac{1}{\pi}\left(\sin x + \sin 2x + \frac{1}{3}\sin 3x + 0\sin 4x + \frac{1}{5}\sin 5x + \dots\right)$$

Additional Problem:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V(x)\psi_1 = E_1\psi_1 \qquad \leftarrow \text{multiply by } \psi_2$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V(x)\psi_2 = E\psi_2 \qquad \leftarrow \text{multiply by } \psi_1$$

Difference:

$$-\frac{\hbar^2}{2m}\left(\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}\right) = (E_1 - E_2)\psi_1\psi_2$$

$$-\frac{\hbar^2}{2m}\frac{d}{dx}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right) = (E_1 - E_2)\psi_1\psi_2$$

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\,\frac{d}{dx}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right) = (E_1 - E_2)\int_{-\infty}^{\infty} dx\,\psi_1\psi_2$$

$$= -\frac{\hbar^2}{2m}\left(\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right)\Bigg|_{-\infty}^{\infty}$$

$$= 0 \quad \text{if } \psi_2(\pm\infty) = \psi_1(\pm\infty) = 0$$

$$\Longrightarrow \boxed{\int_{-\infty}^{\infty} dx\,\psi_1\psi_2 = 0} \quad \text{if } E_1 \neq E_2$$