Phys 475 S'10 Problem Set 4 Solutions

9.5 $(AA^T)^T = (A^T)^T A^T = A A^T$ since $(A^T)^T = A$.

Since AA^T equals its transpose, it is a symmetric matrix.

9.6 $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ symmetric | $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ skew-symmetric

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ real | $\begin{pmatrix} i & 2i \\ 3i & 4i \end{pmatrix}$ pure imaginary

9.11 $H^+ = (\bar{H})^T$. If H is real then $\bar{H} = H$, so $H^+ = H^T$.
If H is Hermitian then $H^+ = H$. Hence, for a real, Hermitian matrix, $H^+ = \boxed{H^T = H}$, so H is symmetric.
If U is unitary then $U^+ U = \mathbb{1}$. If U is also real then $U^+ U = \boxed{U^T U = \mathbb{1}}$, so U is an <u>orthogonal</u> matrix.

9.22 If U is unitary, $[U, U^+] = UU^+ - U^+ U = \mathbb{1} - \mathbb{1} = 0$.
If O is orthogonal, $[O, O^+] = [O, O^T] = OO^T - O^T O = \mathbb{1} - \mathbb{1} = 0$.
If S is symmetric, $[S, S^+] \underset{S\text{ real}}{=} [S, S^T] = [S, S] = 0$

If A is antisymmetric, $[A, A^+] \overset{A\text{ real}}{=} [A, A^T] = [A, -A] = -[A, A] = 0$
If H is Hermitian, $[H, H^+] = [H, H] = 0$
If M is anti-Hermitian, $[M, M^+] = [M, -M] = -[M, M] = 0$.

Hence, any matrix belonging to any of these classes of matrices is <u>normal</u>.

9.23 $(AA^\dagger)^\dagger = (A^\dagger)^\dagger A^\dagger = AA^\dagger$ since $(A^\dagger)^\dagger = A$. ✓

$(A + A^\dagger)^\dagger = A^\dagger + A = A + A^\dagger$. ✓

$(i(A - A^\dagger))^\dagger = -i(A^\dagger - A) = i(A - A^\dagger)$ ✓

using $i^\dagger = -i$

9.25 a) Let $A = O^{-1}$. If O is orthogonal then $A = O^{-1} = O^T$.

$$A^T A = (O^T)^T O^T = O O^T = \mathbb{1}.$$

Hence, $A = O^{-1}$ is an orthogonal matrix.

b) Let $A = U^{-1}$. If U is unitary then $A = U^{-1} = U^\dagger$.

$$A^\dagger A = (U^\dagger)^\dagger U^\dagger = U U^\dagger = \mathbb{1}.$$

Hence, $A = U^{-1}$ is a unitary matrix.

c) If H is Hermitian and U is unitary, then let $A = U^{-1} H U$.

$$A^\dagger = (U^{-1} H U)^\dagger = U^\dagger H^\dagger (U^\dagger)^\dagger = U^{-1} H U = A$$

using $U^{-1} = U^\dagger$ using $H^\dagger = H$.

Hence, $A = U^{-1} H U$ is a Hermitian matrix.

13.17 $\{\text{Real numbers} \setminus 0\}$

not including

1) Closure: Product of nonvanishing real numbers is a nonvanishing real #.

2) Ordinary multiplication is associative.

3) Unit element is 1, which is a nonvanishing real #.

4) Inverse of an element is its reciprocal.

Hence, $\{\text{Real numbers} \setminus 0\}$ w/ ordinary multiplication form a group.

Similarly for $\{\text{Complex numbers} \setminus 0\}$.

For $\{re^{i\theta} \text{ with } r = 1\}$, the analysis is also similar. We need to check closure, which is satisfied because the product of two complex #'s with unit magnitude is another complex # w/ unit magnitude.

13.20 If $\det A = -1$, and $\det B = -1$, then

$$\det(AB) = (\det A)(\det B) = +1.$$

Hence, the set of 3×3 matrices w/ determinant -1 is <u>not closed</u> under matrix multiplization.

Furthermore, the identity matrix, which would be the unit element in the group, is not in the set because $\det \mathbb{1} = +1$.

Hence, {orthogonal 3×3 matrices w/ determinant -1} with matrix multiplication do not form a group.

<u>Comment</u>: In problem 13.17, note that including 0 in the set {Real numbers} eliminates the existence of an inverse, since 0 has no inverse. Hence, {Real numbers} with ordinary multiplication is <u>not</u> a group.