Phys 475 S'10 Problem Set 3 Solutions

4.21 Two vectors are orthogonal iff their dot product vanishes.

Let $\vec{C} = \vec{B}\,|\vec{A}| + \vec{A}\,|\vec{B}|$ and $\vec{D} = \vec{A}\,|\vec{B}| - \vec{B}\,|\vec{A}|$

$$\vec{C}\cdot\vec{D} = \left(\vec{B}\,|\vec{A}| + \vec{A}\,|\vec{B}|\right)\cdot\left(\vec{A}\,|\vec{B}| - \vec{B}\,|\vec{A}|\right)$$

$$= \vec{B}\cdot\vec{A}\;|\vec{A}||\vec{B}| + \vec{A}\cdot\vec{A}\;|\vec{B}||\vec{B}| - \vec{B}\cdot\vec{B}\;|\vec{A}||\vec{A}| - \vec{A}\cdot\vec{B}\;|\vec{B}||\vec{A}|$$

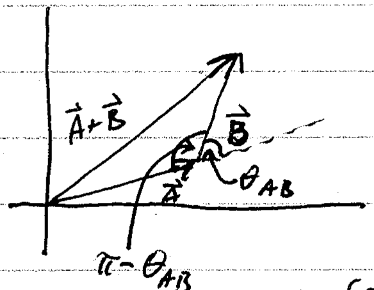
Using $\vec{B}\cdot\vec{A} = \vec{A}\cdot\vec{B}$ the first and last terms cancel.

$\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos(0) = |\vec{A}|^2$

Similarly, $\vec{B}\cdot\vec{B} = |\vec{B}|^2$.

Hence, $\vec{C}\cdot\vec{D} = |\vec{A}|^2|\vec{B}|^2 - |\vec{B}|^2|\vec{A}|^2 = 0$, and $\vec{C}\perp\vec{D}$.

4.22



$$\boxed{(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) =} |\vec{A}|^2 + |\vec{B}|^2 + 2\,\vec{A}\cdot\vec{B}$$

$$= |\vec{A}|^2 + |\vec{B}|^2 + 2\,|\vec{A}|\,|\vec{B}|\cos\theta_{AB}$$

$$= \boxed{|\vec{A}|^2 + |\vec{B}|^2 - 2\,|\vec{A}|\,|\vec{B}|\cos(\pi-\theta_{AB})}$$

$$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = |\vec{A}+\vec{B}|^2$$

We recognize the boxed equation as the <u>law of cosines</u>, where $\pi-\theta_{AB}$ is the angle indicated in the figure.

6.6

$$A^2 = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \mathbb{1}$$

$$B^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \mathbb{1}$$

$$C^2 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \mathbb{1}$$

$$AB = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}i&0\\0&-i\end{pmatrix}, \quad BA = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}-i&0\\0&i\end{pmatrix}$$

$$BC = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&i\\i&0\end{pmatrix}, \quad CB = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} = \begin{pmatrix}0&-i\\-i&0\end{pmatrix}$$

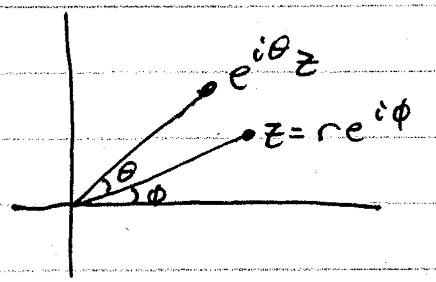
$$AC = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}, \quad CA = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

From these products it immediately follows that

$AB = -BA$, $AC = -CA$, $BC = -CB$, and

$AB - BA = 2iC$, $BC - CB = 2iA$, $CA - AC = 2iB$.

7.19



$$
\begin{aligned}
e^{i\theta}z &= e^{i\theta}(x+iy) \\
&= (\cos\theta + i\sin\theta)(x+iy) \\
&= (\cos\theta\, x - \sin\theta\, y) \\
&\quad + i(\sin\theta\, x + \cos\theta\, y) \\
&= X + iY, \text{ where}
\end{aligned}
$$

$$X = \cos\theta\, x - \sin\theta\, y$$
$$Y = \sin\theta\, x + \cos\theta\, y$$

These two relations are equivalent to:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad ✓$$

Note that $\sin\theta$ and $-\sin\theta$ in the rotation matrix are reversed from our discussion in class. This transformation represents an <u>active</u> rotation of the vector (x, y) by an angle θ. In class we described a <u>passive</u> transformation in which the vector was fixed, but the coordinates rotated by an angle θ, which has an opposite effect on the components of the vector.

9.15 a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad A^{+} = \overline{(A^T)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$ ✓

$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\qquad B^{+} = \overline{(B^T)} = \overline{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = B$ ✓

$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad C^{+} = \overline{(C^T)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = C$ ✓

b) From problem 6.6, $[A, B] = AB - BA = 2iC$,
$[B, C] = 2iA$, $\quad [C, A] = 2iB$

$$
\begin{aligned}
&[A, [B,C]] + [B, [C,A]] + [C, [A,B]] = \\
&= [A, 2iA] + [B, 2iB] + [C, 2iC] \\
&= 2i(AA - AA) + 2i(BB - BB) + 2i(CC - CC) \\
&= 0. \quad ✓
\end{aligned}
$$

9.15 c) $[A,[B,C]] = A(BC-CB) - (BC-CB)A$

$= ABC - ACB - BCA + CBA$

$[B,[C,A]] = B(CA-AC) - (CA-AC)B$

$= BCA - BAC - CAB + ACB$

$[C,[A,B]] = C(AB-BA) - (AB-BA)C$

$= CAB - CBA - ABC + BAC$

Summing these three expressions, we check the Jacobi identity:

$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ ✓

<u>Additional Problems</u>

1. $M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$e^{iM\theta/2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{iM\theta}{2}\right)^n$$

$$M^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

Hence, $M^{2m} = \mathbb{1}$ $\forall$ integer m.

$M^{2m+1} = M$ $\forall$ integer m.

$$e^{iM\theta/2} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{i\theta}{2}\right)^{2n} M^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{i\theta}{2}\right)^{2n+1} M^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} \mathbb{1} + \sum_{n=0}^{\infty} \frac{i(-1)^n}{(2n+1)!}\left(\frac{\theta}{2}\right)^{2n+1} M$$

$$= \cos\left(\frac{\theta}{2}\right)\mathbb{1} + i\sin\left(\frac{\theta}{2}\right) M$$

$$= \cos\left(\frac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & 0 \\ 0 & \cos\theta/2 \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta/2 \\ -\sin\theta/2 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}}$$

2. $M = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = M^{2n} \ \forall$ integer $n > 0$ $\quad (M^0 = \mathbb{1})$

$M^{2n+1} = M \ \forall n$ integer.

$$e^{iM\theta} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\theta)^n M^n$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{(2n)!} (i\theta)^{2n} M^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (i\theta)^{2n+1} M^{2n+1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{i(-1)^n}{(2n+1)!} \theta^{2n+1} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n} & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} & 0 \\ -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

3 a) $\vec{D} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$\vec{D}\cdot\vec{A} = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{A}) - (\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{A})$

$= 0$ using $\vec{A}\cdot\vec{C} = \vec{C}\cdot\vec{A}$ and $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

b) If $\vec{B}$ is parallel to $\vec{C}$ then $\vec{B} = k\vec{C}$ for some real number k.

Then $\vec{D} = (\vec{A}\cdot\vec{C})k\vec{C} - (\vec{A}\cdot(k\vec{C}))\vec{C}$

$= k(\vec{A}\cdot\vec{C})\vec{C} - k(\vec{A}\cdot\vec{C})\vec{C}$

$= 0$

using linearity of the dot product.