Phys 475 S'10 Problem Set 2 Solutions

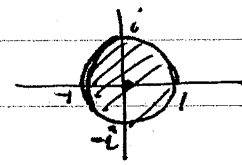
7.9 $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}$

Test convergence by ratio test:

$$\rho_n = \left| \frac{z^{n+1}}{\sqrt{n+1}} \div \frac{z^n}{\sqrt{n}} \right| = \left| \frac{z\sqrt{n}}{\sqrt{n+1}} \right|$$

$\rho = \lim_{n\to\infty} \rho_n = |z|$. Series converges if $\rho < 1$

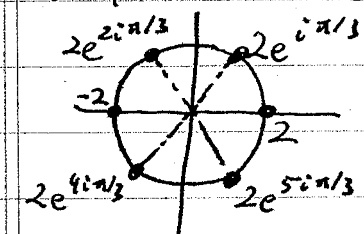
$\rightarrow$ Region of convergence $\boxed{|z| < 1}$



10.6 $64 = 2^6 e^{2\pi i n}$, $n \in$ Integers

$$\sqrt[6]{64} = 2^{6/6} e^{2\pi i n/6} = 2e^{i\pi n/3}$$

$$= 2\left(\cos\frac{\pi n}{3} + i \sin\frac{\pi n}{3}\right) \equiv x_n$$



n	0	1	2	3	4	5
x_n	2	$1+\sqrt{3}i$	$-1+\sqrt{3}i$	-2	$-1-\sqrt{3}i$	$1-\sqrt{3}i$

Hence $\sqrt[6]{64} = \pm 2, \pm 1 \pm i\sqrt{3}$, where the signs are uncorrelated.

10.28 By de Moivre's Theorem, $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i \sin 3\theta$

$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equating the real parts $\longrightarrow$ $\boxed{\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta}$

Equating the imaginary parts $\Longrightarrow$ $\boxed{\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta}$

10.32 $\sqrt[3]{1} = \left(e^{2\pi i n}\right)^{1/3} = e^{2\pi i n/3}$, $n \in$ Integers

$$e^{2\pi i n/3} = \underset{n=0}{1},\ \omega = \underset{n=1}{e^{2\pi i/3}},\ \omega^2 = \underset{n=2}{e^{4\pi i/3}} = \left(e^{2\pi i/3}\right)^2 \checkmark$$

11.10 $\sin(i \ln i) = \sin\left[i\left(\frac{\pi i}{2} + 2\pi i n\right)\right]$, $n \in$ Integers

$$= \sin\left[-\frac{\pi}{2} - 2\pi n\right]$$

$$= \boxed{-1}$$

11.14 $\int_0^{2\pi} \sin^2 4x \, dx = \int_0^{2\pi} \left(\frac{e^{4ix} - e^{-4ix}}{2i}\right)^2 dx$

$$= \int_0^{2\pi} -\frac{1}{4}\left(e^{8ix} + e^{-8ix} - 2\right) dx$$

$$= -\frac{1}{4}\left[\frac{1}{8i} e^{8ix}\Big|_0^{2\pi} - \frac{1}{8i} e^{-8ix}\Big|_0^{2\pi} - 2x\Big|_0^{2\pi}\right]$$

$$= -\frac{1}{4}\left[\frac{1}{8i}\left(e^{16\pi i} - e^0\right) - \frac{1}{8i}\left(e^{-16\pi i} - e^0\right) - 4\pi\right]$$

$$= -\frac{1}{4}\left[\frac{1}{8i}(1-1) - \frac{1}{8i}(1-1) - 4\pi\right]$$

$$= \boxed{\pi}$$

12.20 $\cosh z = \frac{1}{2}(e^{z}+e^{-z})$, $\sinh z = \frac{1}{2}(e^{z}-e^{-z})$

$\Rightarrow e^{z} = \cosh z + \sinh z$, $e^{-z} = \cosh z - \sinh z$

$$e^{nz} = \cosh nz + \sinh nz = (e^{z})^{n} = (\cosh z + \sinh z)^{n}$$

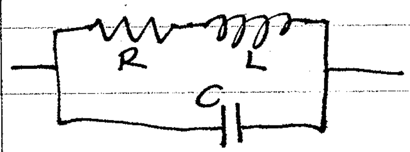
$$e^{-nz} = \cosh nz - \sinh nz = (e^{-z})^{n} = (\cosh z - \sinh z)^{n}$$

$$\Rightarrow \cosh nz = \tfrac{1}{2}(\cosh z + \sinh z)^{n} + \tfrac{1}{2}(\cosh z - \sinh z)^{n}$$

$$\sinh nz = \tfrac{1}{2}(\cosh z + \sinh z)^{n} - \tfrac{1}{2}(\cosh z - \sinh z)^{n}$$

$$\cosh 3z = \tfrac{1}{2}(\cosh^{3} z + 3\cosh^{2} z \sinh z + 3\cosh z \sinh^{2} z + \sinh^{3} z)$$
$$+ \tfrac{1}{2}(\cosh^{3} z - 3\cosh^{2} z \sinh z + 3\cosh z \sinh^{2} z - \sinh^{3} z)$$
$$= \boxed{\cosh^{3} z + 3\cosh z \sinh^{2} z}$$

Similarly, $\boxed{\sinh 3z = \sinh^{3} z + 3\sinh z \cosh^{2} z}$

16.8 

$Z_R = R$, $Z_L = i\omega L$, $Z_C = \frac{1}{i\omega C}$

R, L in series: $Z_{RL} = Z_R + Z_L = R + i\omega L$

RL and C in parallel: $\frac{1}{Z} = \frac{1}{Z_{RL}} + \frac{1}{Z_C}$

$$\frac{1}{Z} = \frac{1}{R+i\omega L} + \frac{1}{1/(i\omega C)} = \frac{1}{R+i\omega L} + i\omega C$$

$$= \frac{1 + (R+i\omega L)i\omega C}{R+i\omega L} = \frac{1-\omega^{2}LC + i\omega RC}{R+i\omega L}$$

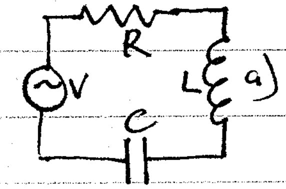
$$Z = \frac{R+i\omega L}{1-\omega^{2}LC + i\omega RC} \cdot \frac{1-\omega^{2}LC - i\omega RC}{1-\omega^{2}LC - i\omega RC} = \boxed{\frac{R + i(\omega L - \omega^{3}L^{2}C - \omega R^{2}C)}{(1-\omega^{2}LC)^{2} + \omega^{2}R^{2}C^{2}}}$$

<u>Resonance</u>: $\operatorname{Im} Z = 0 \Rightarrow \omega L - \omega^{3}L^{2}C - \omega R^{2}C = 0$, $\omega \neq 0$

$$\omega^{2}L^{2}C = L - R^{2}C$$

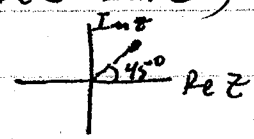
$$\boxed{\omega = \sqrt{\frac{1}{LC} - \frac{R^{2}}{L^{2}}}}$$

16.9



a) $Z = R + i\left(\omega L - \frac{1}{\omega C}\right)$

$\arg Z = 45^\circ \Longrightarrow R = \omega L - \frac{1}{\omega C}$ $(\mathrm{Re}\,Z = \mathrm{Im}\,Z)$



$$\omega^2 LC - \omega RC - 1 = 0$$

$$\omega = \frac{RC \pm \sqrt{R^2C^2 + 4LC}}{2LC}$$

$$= \boxed{\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}}$$

b) Resonant frequency: $\mathrm{Im}\,Z = 0$

$$\omega L - 1/\omega C = 0 \Rightarrow \boxed{\omega = \sqrt{\frac{1}{LC}}}$$

16.12 $\left|\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right|^2 = \left|\sum_{n=0}^{\infty} r^{2n}(\cos n\theta + i\sin n\theta)\right|^2$

$$= \left|\left(\sum_{n=0}^{\infty} r^{2n}\cos n\theta\right) + i\left(\sum_{n=0}^{\infty} r^{2n}\sin n\theta\right)\right|^2$$

$$= \left(\sum_{n=0}^{\infty} r^{2n}\cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n}\sin n\theta\right)^2$$

$$\sum_{n=0}^{\infty} r^{2n} e^{in\theta} = \sum_{n=0}^{\infty}(r^2 e^{i\theta})^n = \frac{1}{1 - r^2 e^{i\theta}} \quad \text{if } r < 1$$

$$\left|\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right|^2 = \frac{1}{(1 - r^2 e^{i\theta})} \cdot \frac{1}{(1 - r^2 e^{-i\theta})}$$

$$= \left(1 + r^4 - r^2 e^{i\theta} - r^2 e^{-i\theta}\right)^{-1}$$

$$= \left(1 + r^4 - 2r^2\cos\theta\right)^{-1}$$

16.13 $y = e^{i\omega t} \longrightarrow \frac{d^2y}{dt^2} = (i\omega)^2 e^{i\omega t} = -\omega^2 y$

$y = e^{-i\omega t} \longrightarrow d^2y/dt^2 = (-i\omega)^2 e^{-i\omega t} = -\omega^2 y$

$y = \cos\omega t \longrightarrow d^2y/dt^2 = -\omega^2 \cos\omega t = -\omega^2 y$

$y = \sin\omega t \longrightarrow d^2y/dt^2 = -\omega^2 \sin\omega t = -\omega^2 y$

17.27 a) $z = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$

$\bar{z} = \mathrm{Re}\, z - i\, \mathrm{Im}\, z$

Solving for Re z and Im z,

$$\boxed{\begin{aligned} \tfrac{1}{2}(z+\bar{z}) &= \mathrm{Re}\, z \\ \tfrac{1}{2i}(z-\bar{z}) &= \mathrm{Im}\, z \end{aligned}}$$

b) $|e^z|^2 = e^{\mathrm{Re}\, z + i\, \mathrm{Im}\, z}\; e^{\mathrm{Re}\, z - i\, \mathrm{Im}\, z}$

$= \boxed{e^{2\,\mathrm{Re}\, z}}$

c) $\left| e^{(1+ix)^2(1-it) - |1+it|^2} \right|^2$

$= \left| \exp\left((1-x^2+2ix)(1-it) - (1+t^2) \right) \right|^2$

$= \left| \exp\left((\cancel{1} - x^2 + 2xt \cancel{-1} - t^2) + i(2x - t + x^2 t) \right) \right|^2$

$= \left| \exp\left[-(x-t)^2 + i(2x - t + x^2 t) \right] \right|^2$

$= \boxed{e^{-2(x-t)^2}}$