Phys 475 S'10 Problem Set 1 Solutions

1.12 Fraction of impurity removed after $N$ stages is

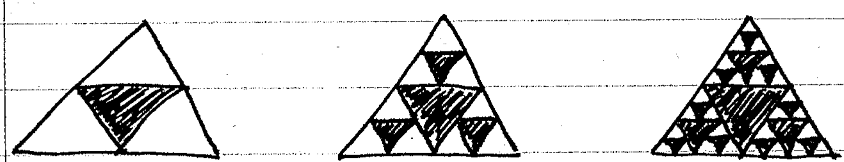
$$S_N = \sum_{m=1}^{N} \left(\frac{1}{n}\right)^m = \frac{1}{n} \frac{\left(1 - (1/n)^N\right)}{1 - 1/n}$$

As $N \to \infty$, $S_N \to \frac{1}{n}\left(1 - \frac{1}{n}\right)^{-1}$

If $n=2$, $S_N \to 1$ so asymptotically all of the impurity is removed.

If $n=3$, $S_N \to 1/2$ so asymptotically only $1/2$ of the impurity is removed.

1.15



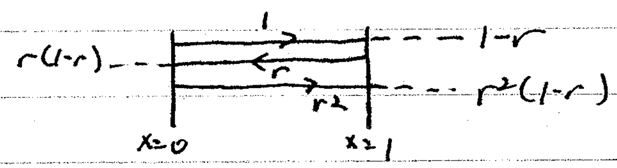
1st iteration  2nd iteration  3rd iteration

Each iteration removes $\frac{1}{4}$ of the remaining area, so after $N$ iterations the remaining area is $A_N = \left(\frac{3}{4}\right)^N$.

As $N \to \infty$, $A_N \to 0$.

The total area removed as $N \to \infty$ is then the entire area $A$.

1.16




Fraction escaping at $x=1$: $f_1 = (1-r) + (1-r)r^2 + (1-r)r^4 + \ldots$

$$= (1-r) \cdot \frac{1}{1-r^2} = \frac{1}{1+r}$$

Fraction escaping at $x=0$: $f_0 = (1-r)r + (1-r)r^3 + (1-r)r^5 + \ldots$

$$= (1-r)r \cdot \frac{1}{1-r^2} = \frac{r}{1+r}$$

Maximum fraction escaping at $x=0$: $r \to 1$, $\boxed{f_0 \to \frac{1}{2}}$

7.9 $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

Let $b_n = a_n + |a_n|$.

If $a_n \geq 0$ then $b_n = 2|a_n|$. Otherwise $b_n = 0$.

$\sum b_n \leq 2 \sum |a_n|$ converges because $\sum |a_n|$ converges.

$$\sum a_n = \sum (b_n - |a_n|) = \underbrace{\sum b_n}_{\text{converges}} - \underbrace{\sum |a_n|}_{\text{converges.}}$$

Hence, $\sum a_n$ is the difference between two convergent series, so it also converges.

9.18 $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{\log n}}$ converges by the alternating series test because $\frac{1}{2^{\log n}} \rightarrow 0$ as $n \rightarrow \infty$.

10.2 $\sum_{n=0}^{\infty} \frac{(2x)^n}{3^n}$. Test convergence by the ratio test.

$$\rho_n = \left| \frac{(2x)^{n+1}}{3^{n+1}} \div \frac{(2x)^n}{3^n} \right| = \left| \frac{2x}{3} \right|$$

$\rho = \lim_{n\to\infty} \rho_n = \left|\frac{2x}{3}\right| < 1$ if $-\frac{3}{2} < x < \frac{3}{2}$, so the sum converges for such values of $x$.

Test the endpoints of the interval:

$x = -3/2 \rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$ not convergent

$x = +3/2 \rightarrow \sum_{n=0}^{\infty} \frac{(3)^n}{3^n} = \sum_{n=0}^{\infty} 1$ not convergent.

Hence, the interval of convergence is $\boxed{-\frac{3}{2} < x < \frac{3}{2}}$.

10.17 $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$

Ratio test: $\rho_n = \left| \frac{(x+1)^{n+1}}{(n+1)} \div \frac{(x+1)^n}{n} \right| = \left| \frac{(x+1)\,n}{(n+1)} \right|$

$$\rho = \lim_{n\to\infty} \rho_n = |x+1| < 1 \text{ for } -2 < x < 0$$

Hence, the series converges in that interval.

Test endpoints:

$x = -2 \longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test

(because $\int^{\infty} dx\, \frac{1}{x}$ diverges)

$x = 0 \longrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test

(because $\frac{1}{n} \to 0$ as $n \to \infty$)

Hence, the interval of convergence is $\boxed{-2 < x \leq 0}$

13.29 $\sqrt{1 + \ln(1+x)} = \sqrt{1 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)}$

$$= 1 + \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) - \frac{1}{8}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)^2 + \dots$$

$$= 1 + \frac{1}{2}x + \left(-\frac{1}{4} - \frac{1}{8}\right)x^2 + \dots$$

$$= \boxed{1 + \frac{1}{2}x - \frac{3}{8}x^2 + \dots}$$

13.39 $f(x) = \sin x = \sin\left(\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right) = \cos\left(x - \frac{\pi}{2}\right)$

$$= \boxed{1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 + \dots}$$

14.3 The Maclaurin series for $\sqrt{1+x}$ is

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

For $x > 0$ this is an alternating series because

$$\binom{1/2}{n} = \frac{1}{n!} \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)$$

alternates in sign as $n$ increases.

We are interested in the region $0 < x < 1/2$, for which successive terms decrease in magnitude, $|a_{n+1}| \leq |a_n|$

For such alternating series with $a_n \to 0$ as $n \to \infty$, the error in keeping $n$ terms is bounded by

$$|S - (a_1 + a_2 + \cdots + a_n)| \leq |a_{n+1}|$$

Hence,

$$\left| \sqrt{1+x} - \left(1 + \frac{1}{2}x\right) \right| \leq \frac{1}{8}x^2$$

For $0 < x < \frac{1}{2}$, the maximum error occurs as $x \to \frac{1}{2}$, for which $\frac{1}{8}x^2 = \frac{1}{32} = \boxed{0.031}$

14.4 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

This is an alternating series with successive terms decreasing in magnitude in the region of interest $0 < x < \frac{1}{2}$.

Hence, $|\sin x - x| \leq \frac{x^3}{3!}$

$$< \frac{1}{6} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{48} = \boxed{0.021}$$

Similarly for $0 < x < 0.1$ the error is bounded by

$$|\sin x - x| \leq \frac{x^3}{3!} < \frac{1}{6}(0.1)^3 = \boxed{0.0002}$$

15.31 Radius of $(n-1)^{th}$ disk $r_n = \frac{1}{n \log n}$, $n \geq 2$.

a) Area of $(n-1)^{th}$ disk $A_n = \pi r_n^2 = \frac{\pi}{n^2(\log n)^2}$

Total area of disks $A = \sum_{n=2}^{\infty} A_n = \sum_{n=2}^{\infty} \frac{\pi}{n^2(\log n)^2}$

Since $0 < \frac{\pi}{n^2(\log n)^2} < \frac{\pi}{n^2}$ for $n \geq 3$

and $\sum_{n=3}^{\infty} \frac{\pi}{n^2}$ converges by the integral test $\left( \int^{\infty} dx \, \frac{\pi}{x^2} = -\frac{\pi}{x} \Big|^{\infty} = 0 \right)$

it follows that $\sum_{n=2}^{\infty} A_n$ converges.

$\Longrightarrow$ Total area of disks is finite.

b) Circumference of $(n-1)^{th}$ disk $C_n = 2\pi r_n = \frac{2\pi}{n \log n}$

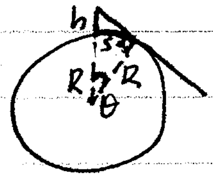
$\sum_{n=2}^{\infty} C_n$ diverges by the integral test:

$I = 2\pi \int^{\infty} dx \, \frac{1}{x \log x}$ Let $x = e^u$, $\frac{1}{x \log x} = \frac{1}{u e^u}$
$dx = e^u \, du$

$I = 2\pi \int^{\infty} du \, \cancel{e^u} \frac{1}{u \cancel{e^u}} = 2\pi \log u \Big|^{\infty}$ diverges

Hence, the total length of wire diverges.

15.33




$$\cos\theta = \frac{R}{R+h} \longrightarrow \sec\theta = \frac{R+h}{R} = 1 + \frac{h}{R}$$

$$\frac{h}{R} = \sec\theta - 1 = \frac{1}{1-\frac{\theta^2}{2!}+\dots} - 1$$

$$= \left(1 + \frac{\theta^2}{2!} + \dots\right) - 1 = \frac{\theta^2}{2}$$

For $h \ll R$, $\frac{h}{R} \approx \frac{\theta^2}{2} \longrightarrow \theta \approx \sqrt{\frac{2h}{R}}$

Distance along the earth $s = R\theta \approx R\sqrt{\frac{2h}{R}} = \boxed{\sqrt{2hR}}$

Earth's radius in miles $R \approx 3950$ mi

$$s(mi) = \sqrt{2\,(3950\text{ mi})\,h(mi)}$$
$$= \sqrt{2\,(3950\text{ mi})\,h(ft)\,\frac{1\text{ mi}}{5280\text{ ft}}}$$
$$\approx \boxed{\sqrt{\frac{3\,h(ft)}{2}}}$$

16.3 $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges by the integral test $\left(\int^{\infty} dx\,\frac{1}{x^{3/2}} \text{ converges}\right)$

$$\sum_{n=2}^{\infty} \frac{1}{n^{3/2}} = \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{27}} + \frac{1}{\sqrt{64}} + \frac{1}{\sqrt{125}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{3n} = \sum_{n=1}^{\infty} \frac{1}{(9n^2)^{1/2}} = \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} + \frac{1}{\sqrt{81}} + \frac{1}{\sqrt{144}} + \dots$$

For large $n$, $\frac{1}{n^{3/2}} < \frac{1}{3(n-1)}$ so even though the first few terms in $\sum\frac{1}{n^{3/2}}$ are larger than the first few terms in $\sum\frac{1}{3n}$, the former series converges while the latter diverges.