

An Analysis of Optical Quantum Noise in Coherent Interactions for a Three-level Atom

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1 Abstract

Interaction of light with atoms creates long-lived atomic coherences, and can modify the quantum properties of original optical fields (such as squeezing of light) and sometimes lead to the generation of new quantum beams. We would like to develop the proper calculational techniques to describe the quantum treatment of light-atom interactions in the process of electromagnetically induced transparency (EIT) and four-wave mixing (FWM) in a three-level Λ system. More specifically, the goal is to understand the formalism behind applying the Langevin forces to calculate both mean values and quantum fluctuations of the system beyond a noiseless approximation. We were able to apply the Langevin forces to a two-level atomic system, and generate noise functions that behave realistically when compared to experiments involving a ^{87}Rb atomic system. We were also able to generate noise functions for a three-level system, which we wish to verify computationally and experimentally.

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2 Introduction

2.1 Semi-classical Description of a Two-level System

We begin with a basic schematic of the interaction of light with an atomic medium. This relationship can be modeled in many different ways. Here, we take a semi-classical approach where the incoming radiation is described by an oscillatory perturbation given by the Hamiltonian,

$$\hat{H}_p = -\hat{\mu}Ee^{-i\omega t} \quad (1)$$

where $\hat{\mu}$ is the dipole moment and \hat{E} is the amplitude of the electromagnetic field. We consider this perturbation as it acts on a two level system $|1\rangle, |2\rangle$ where $\omega_0 = \omega_2 - \omega_1$ is the optical frequency between the states, see Fig.1. It is possible to write the wavefunction of this system as a linear combination of stationary states,

$$\psi(r, t) = C_1(t)e^{-i\omega_1 t}|1\rangle + C_2(t)e^{-i\omega_2 t}|2\rangle \quad (2)$$

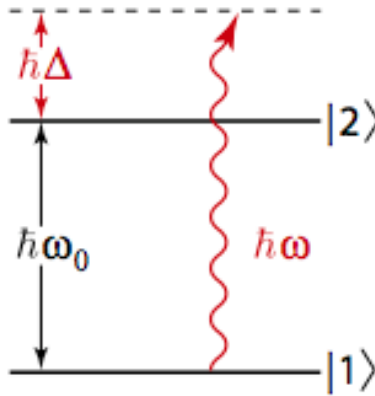


Figure 1: Diagram of Two-Level system where the ground state is $|1\rangle$ and the excited state is $|2\rangle$, separated by an energy $\hbar\omega_0$

Let the system begin in state $|1\rangle$. Then, after solving the time dependent Schrodinger equation with the total Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_p \quad (3)$$

where $\hat{H}_0|i\rangle = \hbar\omega_i|i\rangle$ is the free atom energy, The probability of the system, given by the coefficients $|C_1|^2$ and $|C_2|^2$, to be in either state $|1\rangle$ or $|2\rangle$ also oscillates with time, see Fig. 2. The relevant frequencies that dictate the probabilities depend on the amplitude of the electric

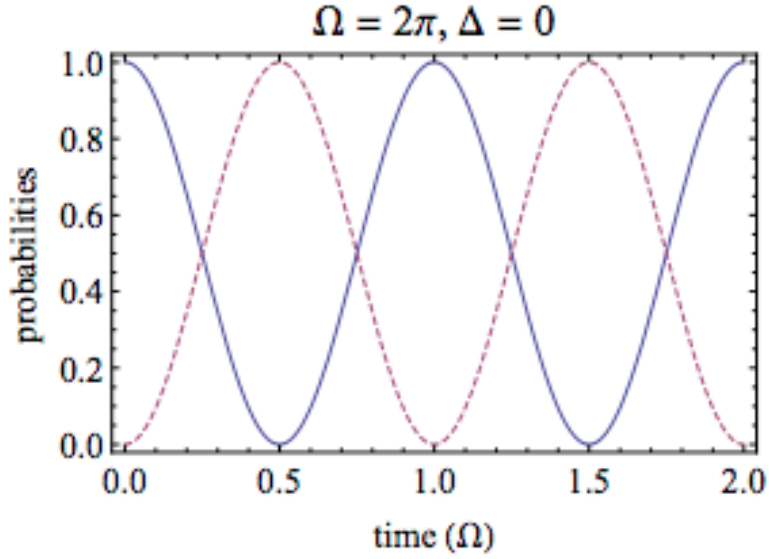


Figure 2: Semiclassical model: oscillation at the Rabi Frequency of the population of a Two-level system

field ($\Omega = \frac{2\mu E}{\hbar}$) and the detuning frequency $\Delta = \omega - \omega_0$. However, this model of the field-atom interaction is incomplete. It does not describe certain phenomena such as spontaneous emission and quantum fluctuations of light [3]. To account for these processes, we move to a quantum mechanical description of both the atom and field.

2.2 Quantization of the Field

In the next step, we extend the treatment of a two-level system for a fully quantum electromagnetic field. To quantize the electromagnetic field, we decompose the field into its modes of oscillation and represent each mode as a simple harmonic oscillator. This method will allow us to express the Hamiltonian of the field with the simple harmonic oscillator annihilation and creation operators,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad (4)$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (5)$$

where the electric field vector can be expressed as

$$\hat{E} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} (\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \quad (6)$$

We use this formalism to describe a set of states of the simple harmonic oscillator (and thus electromagnetic radiation) that best minimizes the Heisenberg uncertainty relationship for amplitude and phase. By using this relationship in the Hamiltonian of the harmonic oscillator,

we find that the states that satisfy this restriction are the eigenstates of the lowering operator (See Eq. 4). These states are known as the coherent states, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. However, since \hat{a} and \hat{a}^\dagger do not commute, the amplitude and phase of the electromagnetic field cannot be measured simultaneously resulting in fundamental quantum noise. Though, it is possible to decrease the uncertainty of either the system's amplitude or phase, while remaining within the bounds of the Heisenberg uncertainty principle. These states are known as squeezed coherent states [3].

Consider again a two-level system modeled by the atomic levels, $|1\rangle$ and $|2\rangle$. For these states, we can write a collection of slowly varying operators averaged over their position in the z direction. These are known as the coherence operators,

$$\hat{\sigma}_{uv}(z, t) = |u\rangle\langle v|e^{(-i\omega_{uv}+ik_{uv}z)} \quad (7)$$

We will express the Hamiltonian of this system with the coherence operators and obtain an approximation for the system that will later allow us to incorporate the Langevin operators that describe the incoherent processes to find the mean value and noise of the applied field. Then, we will use a similar process to analyze a three-level Λ system, though simplified to two ground states and one excited state.

3 Motivation

3.1 Nonphysical Prediction due to Incomplete Account of Spontaneous Emission

When modeling the interaction between an external field, \hat{a} and a two-level atomic system (characterized by the density operator $\hat{\sigma}_{21}$), we begin with the propagation equation

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right) = ig_a N \hat{\sigma}_{21} \quad (8)$$

where g_a is the atom-field coupling constant, and N is the number of atoms in the system. The steps to solve this differential equation are given in much greater detail in Section 5. We present only general steps here. First, we find $\hat{\sigma}_{21}$ by asserting the Hamiltonian and equations of motion of a two-level system. Then, we can substitute the solution, proportional to \hat{a} , into Eq. 8. By applying a Fourier transform to operate in the frequency domain, we can solve a differential equation to find the behavior of the field, \hat{a} as it propagates through space. This can be mathematically expressed as

$$\hat{a}_{out} = \hat{a}_0 e^{-\alpha z} \quad (9)$$

where the output field operator, a_{out} is decaying over space, z , proportional to the constant α . To explain this, consider the quantum mechanical effect of spontaneous emission. First,

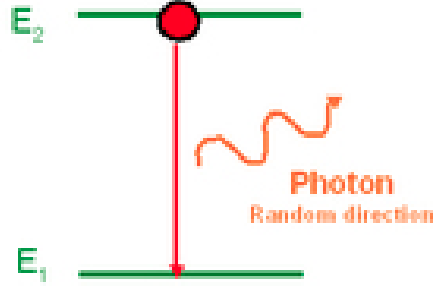


Figure 3: Spontaneous emission: a particle decays from the atom’s excited state without the influence of the field

photons from the laser field are absorbed to promote atoms to the excited state. However, the atoms decay spontaneously, leading to a loss of energy in the original beam as light propagates through the absorbing material, see Fig. 3. However, we know that the commutation relations must hold in a closed system where

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad (10)$$

Consider

$$[\hat{a}_{out}, \hat{a}_{out}^\dagger] = [\hat{a}_{in}e^{-\alpha z}, \hat{a}_{in}^\dagger e^{-\alpha z}] = e^{-2\alpha z}[\hat{a}_{in}, \hat{a}_{in}^\dagger] = e^{-2\alpha z} \quad (11)$$

Therefore, the value of the commutation relations will decay over space if we include the known effect of spontaneous emission, in violation of the commutation relations. To account for this loss, we propose that the vacuum state of the field interacts with the atomic system creating a restorative noise generated by the “Langevin Forces.” This interaction repopulates the system. A detailed description and approach is outlined in later sections. Generally, the motivation of my thesis is to find an analytic solution for the noise generated by the field-atom interaction in order to understand (1) the effect of the Langevin Forces on repopulation of field-atom systems in both the two-level and three-level case (2) if our solution agrees with experiments that will be carried out in our lab using laser light and a Rubidium cell.

In the next section, we briefly outline an experiment carried out in our labs in which the absence of including the Langvin forces in the schematic leads to an unrealistic loss of energy for the system.

3.2 Analysis of Quantum Noise in Current Experiments

Experiments in our lab use Four Wave Mixing (FWM) to generate highly correlated nonclassical beams, which can be affected by fluctuations in the field.

In this process, a three-level optical system is pumped with frequency Ω . Two other beams are generated from this process, the seed and conjugate. It is possible to have a highly correlated intensity relationship between the intensity of these beams, as well as an anti-correlated phase relationship. To fully account for these effects, we will assert the Heisenberg-Langevin formalism, developed throughout the rest of this paper, which can be used to analyze system past a noiseless approximation.

The Langevin equations can be thought of as quantum noise operators that account for the the dissipation of energy from the system to the environment, though they originally developed for classical mechanics (see Section 4.1). To begin the analysis, we write the Hamiltonian for a two-level system

$$\hat{H} = \omega_0 \hat{\sigma}_{22} + g_a \hat{a}(z, t) \hat{\sigma}_{21} + H.C. \quad (12)$$

where g_a is the atom-field coupling constant and δ is the atomic frequency between state $|1\rangle$ and $|2\rangle$. We will incorporate the Langevin equations into the atomic term $\hat{\sigma}_{21}$. These forces will account for the dissipation of the field term $\hat{a}(z, t)$. Otherwise, the annihilation operators would decrease the energy of the system in violation of the commutation relations, as mentioned in Section 3. In the rest of this paper, we focus on introducing the Langevin forces and how to incorporate them into light-atomic interactions after proceeding from the Hamiltonian.

4 Methods

4.1 Introduction to Langevin Forces: Brownian Motion

In classical mechanics, Brownian motion is the random movement of particles in a fluid that results from their collision with other atoms or molecules in the medium. If you consider a particle's movement through a fluid, the momentum of the particle will decrease over time, or $\langle p \rangle = \langle p_0 \rangle e^{-\alpha t}$. However, as the particle moves, kinetic energy is transferred to the surrounding environment. This energy will cause the fluid atoms or molecules to oscillate, which will in turn collide with the particle in question, and cause the particle to fluctuate. Energy is fed back into the system. This process is called Brownian motion, see Fig. 4. Here, the momentum will not dissipate to zero. Rather, there will be fluctuations in the motion, and the standard deviation of these fluctuations (Δp) will be proportional to a constant called the "Diffusion Coefficient", D , an important quantity for later analysis. The Langevin forces are used to describe Brownian motion with the classical Langevin equation

$$\frac{d}{dt} p(t) = -\gamma p(t) + F(t) \quad (13)$$

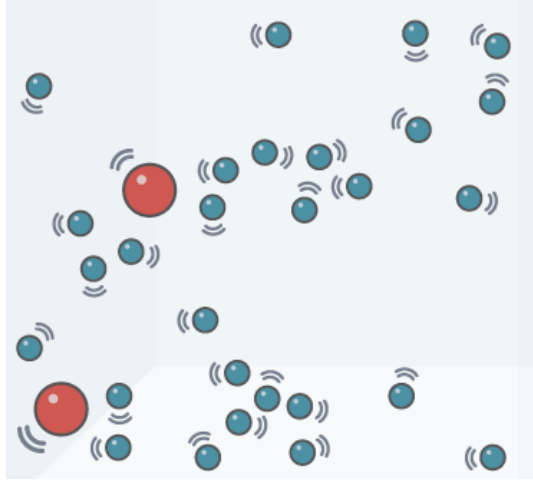


Figure 4: Illustration of Brownian motion. The momentum of the red particle is fluctuating due to collisions with the blue fluid particles

where $-\gamma$ represents the decaying momentum of the particle, and $F(t)$ represents the restorative Langevin force resulting from particle collisions, such that the momentum of the particle does not decay to zero. We can make an analogy to the atomic system. (1) The total momentum of the classical case is decaying similar to that of the field operator due to spontaneous emission and (2) the motion of particle is erratic, similar to the noise of the atomic system due to interaction between the field and the atomic system. Mathematically, the quantum Langevin equation can be stated as

$$\frac{d}{dt}\hat{a}(t) = -\gamma\hat{a}(t) + F(t) \quad (14)$$

Additionally, the average value of the Langevin force is zero, because it is chaotic,

$$\overline{F(t)} = 0 \quad (15)$$

However, since $F(t)$ is correlated with itself, the average value of the product will not be zero. This value is described by

$$\overline{F(t)F(t')} = 2Dg(t-t') \quad (16)$$

where D represents a diffusion coefficient, which is unique to the system of interest and increases over time. In the next system, we show how to calculate D for a two-level system.

4.2 Reservoir and Atomic System

The simplest system that can be assessed using the atomic Langevin equations is the Reservoir problem. We initially focused on this problem to understand the methodology behind including the noise operator in our system, and making the connection between the Langevin

forces and the diffusion coefficients. In this example, we consider a main mode of interest, modeled as a harmonic oscillator, coupled to a reservoir of many harmonic oscillators. The Hamiltonian for this picture is

$$H = H_{main} + H_{res} + H_{int} = \hbar\Omega\hat{a}^\dagger\hat{a} + \sum_j\hbar\omega_j\hat{b}_j^\dagger\hat{b}_j + \sum_j\hbar(g_j\hat{a}^\dagger\hat{b}_j + g_j\hat{b}_j^\dagger\hat{a}) \quad (17)$$

where \hat{a} is the field operator for the main mode, each \hat{b}_j is the field operator for the reservoir states, and g_j describes the interaction between them. We can find equations of motion for each of these variables, where

$$\frac{d}{dt}\hat{a} = \frac{i}{\hbar}[H, \hat{a}(t)] = i\Omega\hat{a}(t) - i\sum_j g_j\hat{b}_j(t) \quad (18)$$

and

$$\frac{d}{dt}\hat{b} = \frac{i}{\hbar}[H, \hat{b}(t)] = i\sum_j(\omega_j\hat{b}_j(t) - ig_j\hat{a}(t)) \quad (19)$$

Next, we perform several mathematical steps to isolate the quantity of interest. We integrate Equation 19 and insert into Equation 18, and remove rapid variations in the system by defining $a(t) = A(t)e^{-i\omega t}$. $A(t)$ now represents the main mode without rapid fluctuations. Again, we skip several formal steps here, but follow the process of [5] closely. By using the form of the quantum Langevin equation (Eq. 14) where $F(t) = i\sum_j g_j\hat{b}_j(t_0)e^{i(\Omega-\omega_j)(t-t_0)}$, and assuming that the frequencies of the reservoir oscillators are closely spaced, and several other considerations, we obtain that

$$\langle F^\dagger(t)F(t) \rangle = 2\langle D_{A^\dagger A} \rangle \delta(t-t') \quad (20)$$

where, according to the Einstein relation, derived using known results of random processes

$$\langle D_{A^\dagger A} \rangle = \frac{d}{dt}\langle A^\dagger(t)A(t) \rangle + \gamma\langle A^\dagger(t)A(t) \rangle \quad (21)$$

The goal of this section is to calculate relevant quantities, such as the equations of motion and diffusion coefficients, for the atomic portion of the atom-light interaction in a two-level system. These will be required when finding the noise of the system as a whole. The initial steps of understanding the atomic portion of the two-level case, explained next, are similar to the steps outlined here. We will begin with the Hamiltonian and find the equations of motion for each atomic variable of interest.

4.3 Two-level System: Calculation of Diffusion Coefficients

In this section, we consider a two-level system and show the general process for finding the diffusion coefficients, and by a direct relationship, the noise operators. Let $|b\rangle$ denote the excited state and $|a\rangle$ denote the ground state. Define the atomic density operators

$$S_+ = |b\rangle\langle a| \quad (22)$$

$$S_- = |a\rangle\langle b| \quad (23)$$

$$S_z = \frac{1}{2}[|b\rangle\langle b| - |a\rangle\langle a|] \quad (24)$$

Using the same steps outlined in Section 4.2, we can find the equations of motion of each atomic variable in terms of the Langevin force operators. We present one here

$$\frac{d}{dt}S_+(t) = -(i\delta + \frac{\gamma}{2})S_+(t) - i\Omega S_z(t) + F_+(t) \quad (25)$$

where γ is the spontaneous emission rate from state $|b\rangle$. Using an extension of the Einstein relation mentioned in Section 4.2 given in [4] where

$$2D_{++} = \frac{d}{dt}\langle S_+^\dagger S_+ \rangle - \langle (\frac{d}{dt}S_+^\dagger)(S_+) \rangle - \langle (S_+^\dagger)(\frac{d}{dt}S_+) \rangle \quad (26)$$

Noting that $\langle S_i F(t) \rangle = 0$ where $i = +, -, z$ because they are anti-correlated, it can be shown that $2D_{++} = \Gamma$, where Γ is the spontaneous emission rate. We can follow an identical process for the other atomic operators.

5 Results for Two Level System: Calculation of Noise Spectra

Our long term goal is to calculate the noise spectra of a three-level system, which is a model for the realistic interaction between laser light and Rb vapor. Our next step in that process is to solve the noise spectra for a two-level system. This will develop the methodology for a more complicated system in the future. It is important here to note that several considerations must be made about both the field and atomic medium before accurately describing the system.

First, we must move from a single-atom to ensemble average of the atomic medium. The number of particles that affect the noise of the field is directly proportional to the fraction of the cell the field propagates through, namely

$$N_i = \left(\frac{z}{L}\right)N \quad (27)$$

Where N_i is the number of particles that influence the noise over a fraction of the cell, $\frac{z}{L}$. Then, we can express the behavior of the ensemble average, S_+^{en} , as

$$S_+^{en}(z, t) = \frac{1}{N_i(z)} \sum_{i \in N_i} S_+^i(z, t) \quad (28)$$

For purposes of notation, we simplify $S_+^{en}(z, t) = S_+$ in following calculations.

Next, we must also treat the field, a running wave, as variable through space. In the case of a running wave, we normalize the value of the commutation relationship to the following

$$[\hat{a}^\dagger(z, t), \hat{a}(z', t')] = \left(\frac{c}{L}\right)\delta(z - z')\delta(t - t') \quad (29)$$

This accounts for the variability of \hat{a} over space.

We begin the mathematical steps to solve the noise by considering the differential equation that describes the propagation and temporal evolution of the quantum field operators in the two-level case

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\hat{a} = igNS_+ \quad (30)$$

where $\hat{a}(z, t)$ describes the field operator and \hat{S}_+ describes the atomic operator averaged over $|b\rangle\langle a|$ in each two-level atomic system, as given by Eq.28. Our goal is to find a relationship that describes $a(\omega)$ in terms of constants and known variables. First, note that

$$\frac{\partial}{\partial t}\hat{S}_+ = -(i\delta t + \frac{\Gamma}{2})\hat{S}_+(t) + ig\hat{a}S_z + F_+(t) \quad (31)$$

Here, the treatment of the Langevin force term, $F_+(t)$, must also be averaged over space. Thus,

$$\langle F^\dagger(z, t)F(z, t) \rangle = \frac{L}{N}\langle D_{A^\dagger A} \rangle \delta(t - t')\delta(z - z') \quad (32)$$

Using the Fourier transform of Eq. 31 to find the equivalent description in the frequency domain, we find that

$$\frac{\partial}{\partial z}\hat{a}(z, \omega) = -[A(\omega)]\hat{a}(z, \omega) + [B(\omega)]F_+(z, \omega) \quad (33)$$

where

$$A(\omega) = \frac{1}{c}\left(\frac{g^2N}{-2i\omega + \Gamma} + i\omega\right) \quad (34)$$

and

$$B(\omega) = \frac{igN}{ic\omega + \frac{c\Gamma}{2}} \quad (35)$$

Solving this differential equation for $\hat{a}(\omega)$, the variable of interest, we obtain that

$$\hat{a}(z, \omega) = \hat{a}(0, \omega)e^{-A(\omega)z} + B(\omega)\int_0^z e^{A(\omega)(z-z')}F(\omega, z')dz' \quad (36)$$

This is an important result, because it describes the behavior of the field as it propagates through the atomic medium over time. First, the mean-field average, $\langle \hat{a} \rangle$ reproduces the modification of the classical field propagation $\langle a(\hat{z}) \rangle = \langle \hat{a}(0) \rangle e^{-A(\omega)L}$. This modification can be quantified, and is a value known as optical depth (OD). It describes the fraction of the mean value of the field that is absorbed by the atomic medium over time. In our case, by ignoring the Langevin forces in the noise equation, we obtain how our field propagates through space.

We proceed by using this relationship, as well as those equations that describe the atomic variable $S_+(\omega)$ to find an element of the noise spectra for the two-level system.

The noise spectra represents the measurable noise of the system. Consider the field operator, $\hat{a}(\omega)$. As a quantized value, we can fully represent $a(\omega)$ as a sum of its average value $\langle \hat{a}(\omega) \rangle$ and fluctuations from the average value, $\delta\hat{a}(\omega)$. These fluctuations capture the quantum

nature of the electromagnetic field. Moreover, we can use Equation 36 to rewrite the noise components in terms of known values as such

$$\delta\hat{a} = \delta\hat{a}(0)e^{-A(\omega)L} + \int_0^z f(\omega, z')dz' \quad (37)$$

where $\int_0^z f(\omega, z')dz' = B(\omega) \int_0^z e^{A(\omega)L} F(\omega, z')dz'$. Define the measurable noise of the system by

$$\langle \delta x(\omega) \delta x^\dagger(\omega) \rangle \quad (38)$$

where $\delta x(\omega) = \delta\hat{a} + \delta\hat{a}^+$. We proceed by substituting Equation 37 into Equation 38 and simplifying. We ignore terms of the product such as $\langle \delta\hat{a}(0)F(\omega) \rangle$ since they are not correlated, thus the product of their average values will be zero.

Then, by making substitutions for $\hat{F}(\omega)$ by the known diffusion coefficients, we find a general form of the measurable noise in a two-level system

$$\langle \delta x(\omega) \delta x^\dagger(\omega) \rangle = e^{-(A(\omega)+A(\omega^*)L} - \frac{2\Gamma}{Nz} \frac{|B(\omega)|^2}{2Re(A(\omega))} (1 - e^{-(A(\omega)+A(\omega^*)L}) \quad (39)$$

Note that this result is a work in progress. We believe the general form to be correct, where the first term represents the system without the effect of the Langevin forces, similar to Equation 9, and the second term represents the restoration of the Langevin force. If we consider the case where the field propagates through the medium for a large distance, we see that $\langle \delta x(\omega) \delta x^\dagger(\omega) \rangle$ will still not decay to zero, a promising conclusion.

Additionally, we plot Eq. 39 to illustrate the behavior of the two-level system noise as a function of frequency. We substitute realistic quantities for the relevant atomic constants, such as $\Gamma \approx 38 \times 10^6 \text{ rad/s}$ and $z = 0.075m$. We show this relationship at two atomic densities, $N = 10^{10}$ and $N = 10^4$ to demonstrate the contribution of the Langevin Forces. We hypothesize that at higher atomic densities, the Langevin forces will have a larger effect, since this noise is related to the number of atoms interfering with the system after spontaneous emission. We use a reference resonant frequency of $5 \times 10^7 \text{ Hz}$. This frequency is hypothetically where the atomic medium absorbs a large fraction of the incoming field, leading to more spontaneous emission and therefore a projected larger Langevin noise contribution. In Fig. 5 where $N = 10^{10}$ we see that at resonance, the Langevin term is indeed the leading contributor to the noise level. However, in Section 6, we will compare these theoretical results to experiments that modulate these trends for a two level system at specific transitions of Rubidium vapor.

As discussed earlier, OD is a quantity of interest in this schematic. It represents the amount of the mean value of the field that is absorbed by the atomic medium over time. Here, $OD = (-A + A^*)L$, as this value dictates the decay of the original field. In Fig. 7 we plot OD as a function of frequency for an atomic density of $N = 10^{10}$. The value of optical depth is smaller than expected. We believe this to be because of a misplaced constant. In this vein, it is important to note that our function of the noise of a two-level system may not fully

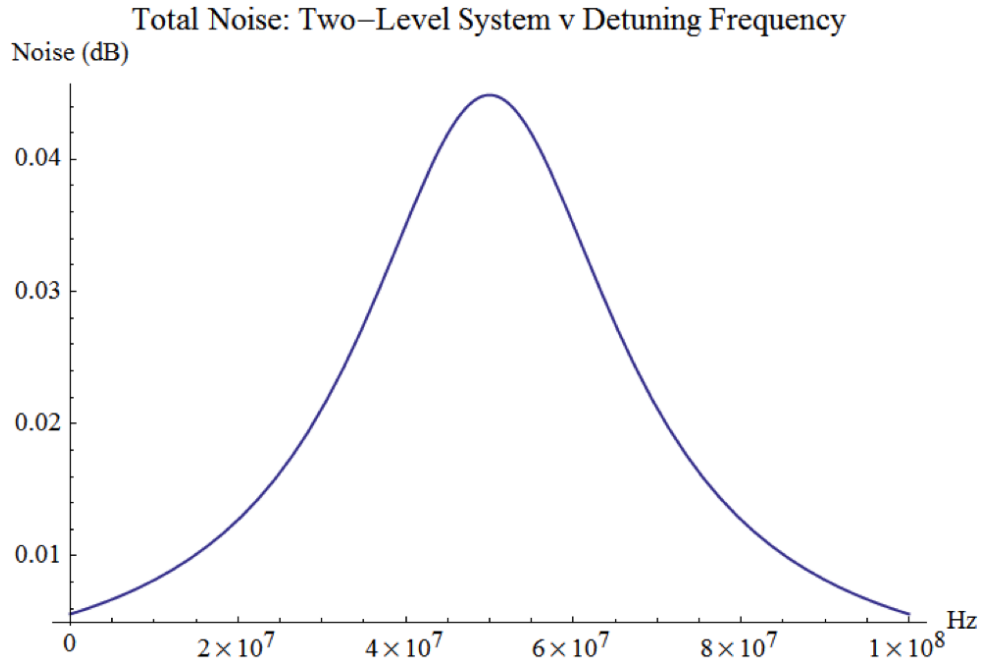


Figure 5: The total noise when the density of the atomic system is high ($N = 10^{10}$). Due to the Langevin term, the noise is relatively large at resonance, $\approx 0.05dB$

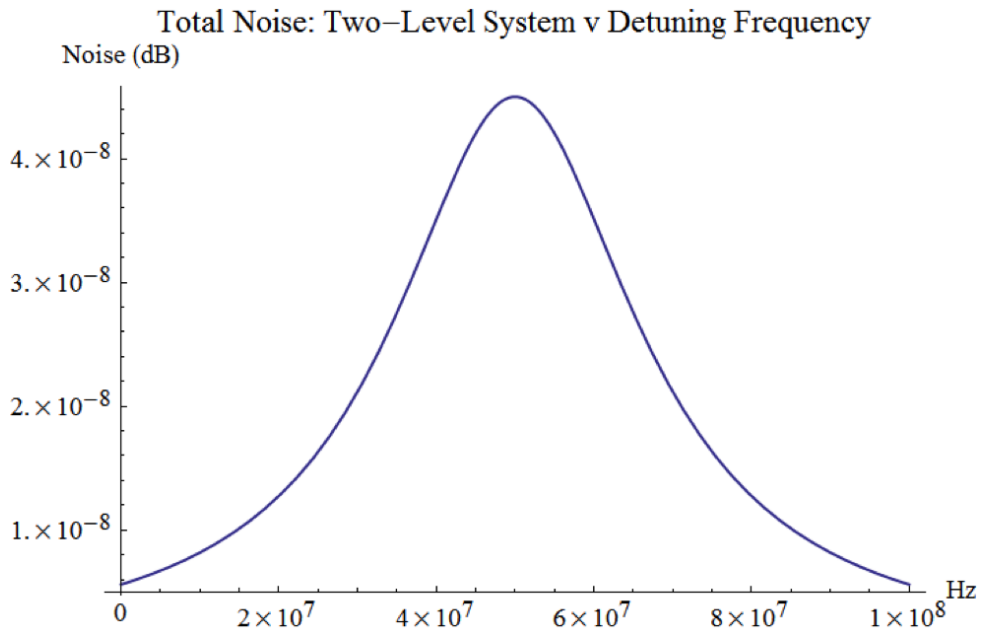


Figure 6: The total noise when the density of the atomic system is low ($N = 10^4$). Due to the Langevin term, the noise is relatively large at resonance, $\approx 5 \times 10^{-8}dB$

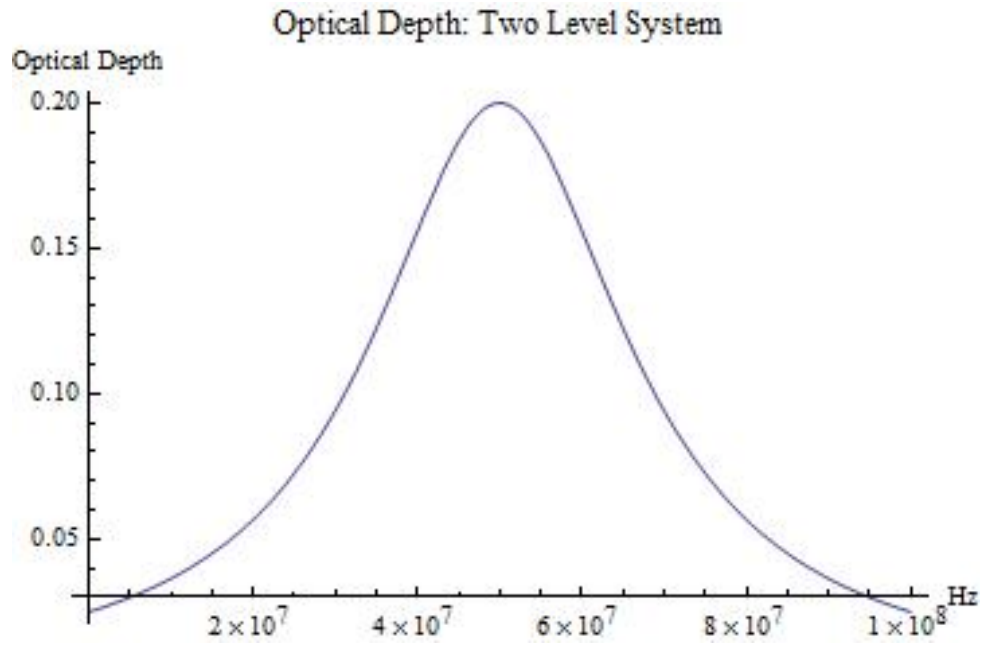


Figure 7: Optical depth versus frequency for two level system

encompass the constants involved. It is still a work in progress. One of our goals is to check this result by verifying the commutation relationship with the value of $\hat{a}(\omega)$ in Equation 36.

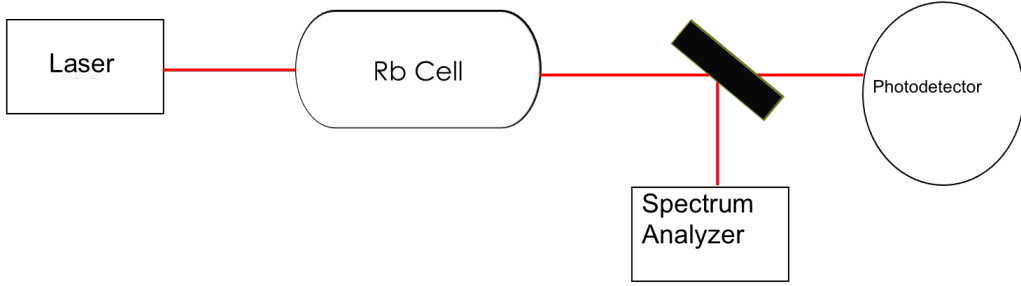


Figure 8: Experimental set up to measure noise and necessity of the Langevin construct

6 Comparison with Experiments

We conducted a basic experiment to confirm the necessity of the Langevin forces in our theoretical work when modeling the interaction between light and a two-level system, and then compare these results to those found in Section 5. As we will discuss in Section 7.1, we use the transitions in Rubidium due to hyperfine splitting between $2 \rightarrow 1'$ and $2 \rightarrow 2'$, and treat these as two distinct two-level systems. The set up consisted of a laser, a heated Rubidium cell, a photodetector, and a spectrum analyzer. In this set up, the frequency of the laser can be shifted by adjusting the length of the optical cavity in which it propagates. One of the cavity mirrors is on a piezoelectric driver, which changes length based on an input voltage. Therefore, by changing the voltage applied to the piezoelectric driver, we were able to control the frequency of the laser. Moreover, the rubidium vapor cell is encapsulated with a thermistor, which allows the temperature of the cell, directly related to the atomic density, to be variable.

Our process was to send the light, or field, through the heated Rb cell, and measure both the total output and noise at varying frequencies. The system reaches resonance at certain frequencies. This is where energy is absorbed by the atomic system and particles move from the ground state to excited state. Thus, Rb absorbs up to 75% of the initial beam, and the total output decreases sharply. At these steep transitions, one would expect the output noise to decrease as well since the noise is generally directly related to the mean output level. However, in this case, the noise increases dramatically, see Fig. 11. Though this may be partially due to other effects such as Doppler broadening, this level of increase in the noise during absorption can be best explained by the Langevin formalism.

In this schematic, when energy is absorbed from the field, for example at detuning frequencies 0 and 0.76 Hz by the atomic system at 90° , and particles rise to the excited state, some are spontaneously emitted. These particles continue to interact with the system, leading to increased noise in the output field. In the following sections, we proceed to theoretically calculate the value for the noise, and compare the general form with the theoretical results.

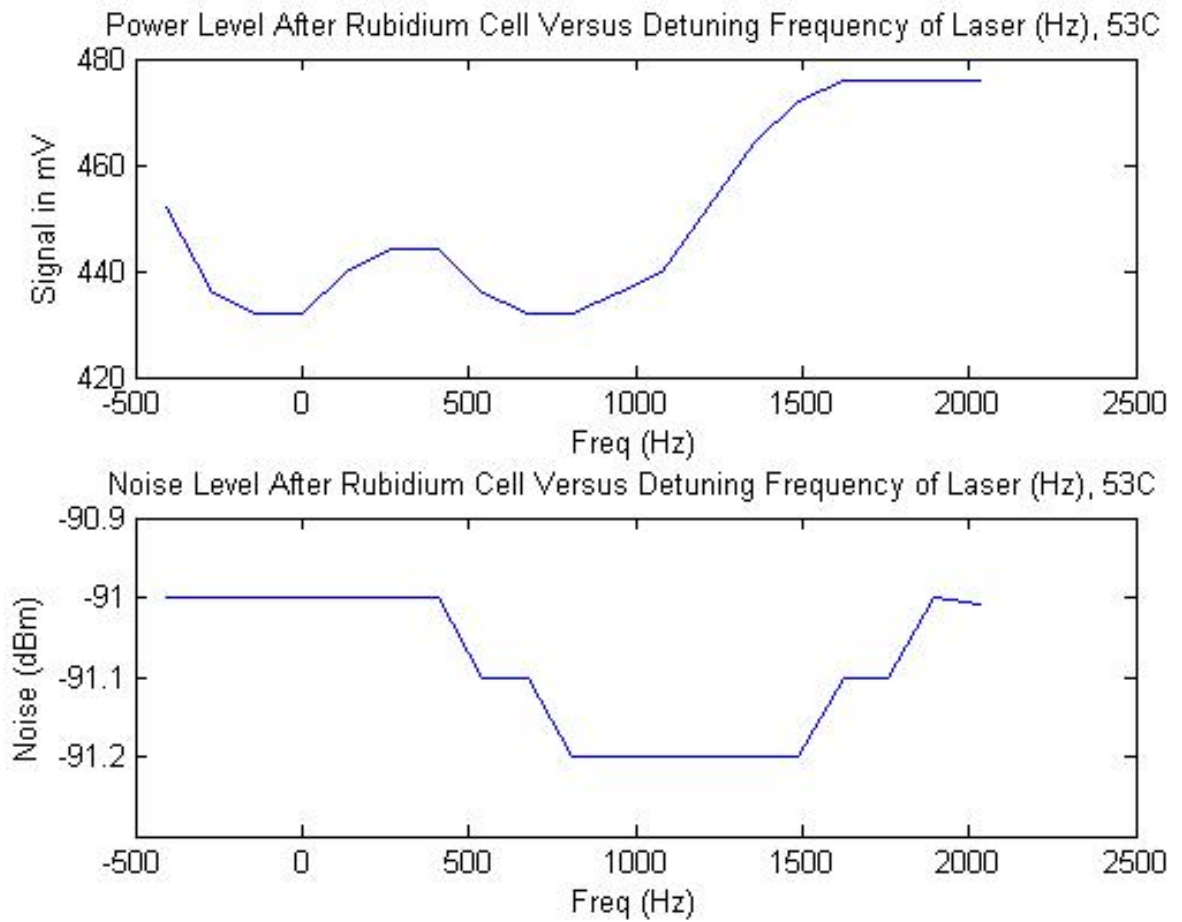


Figure 9: Power and noise of laser after propagating through a cell of ^{87}Rb at 53° . Here, the behavior of the noise is not correlated with trends of the initial signal

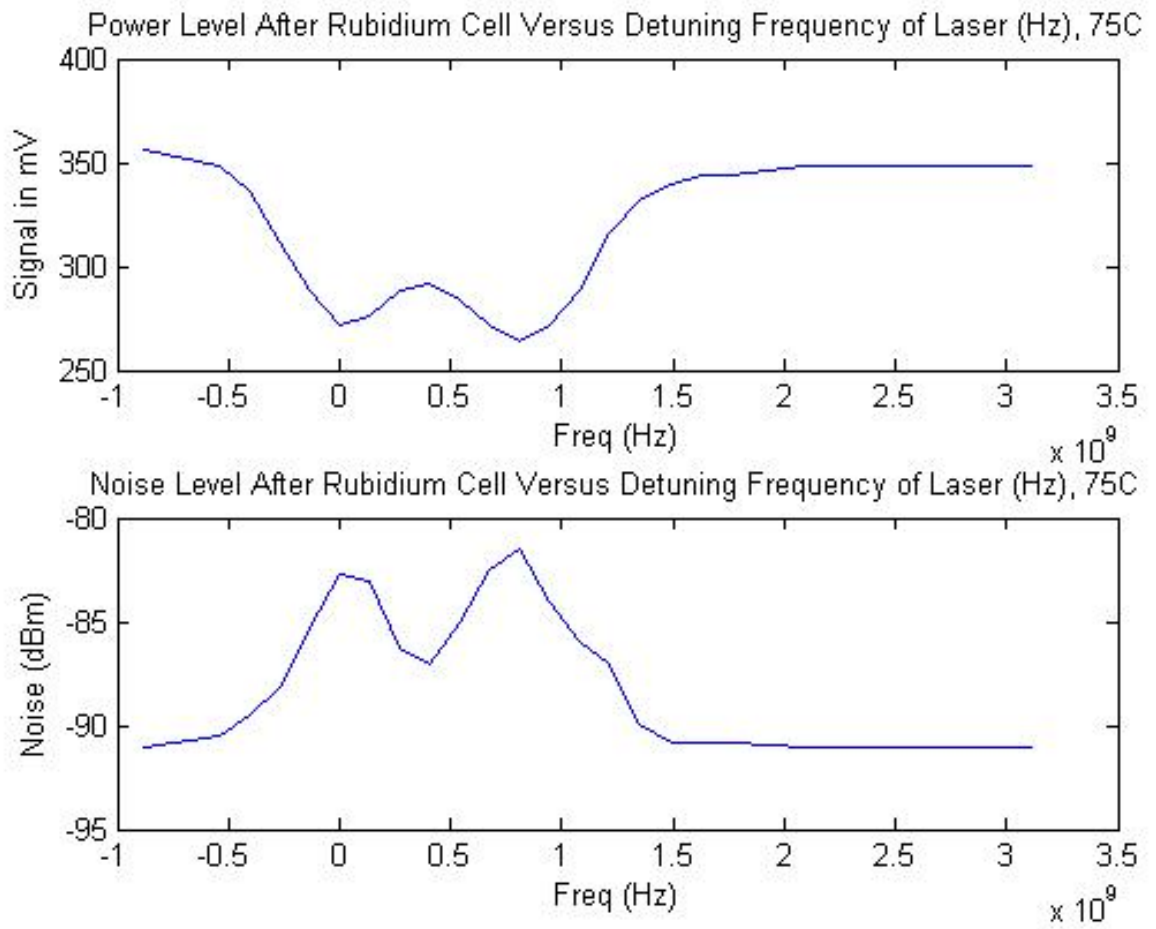


Figure 10: Power and noise of laser after propagating through a cell of ^{87}Rb at 75° . With a higher density than the vapor at 53° , it is possible to see an inverse trend between the initial signal and noise level

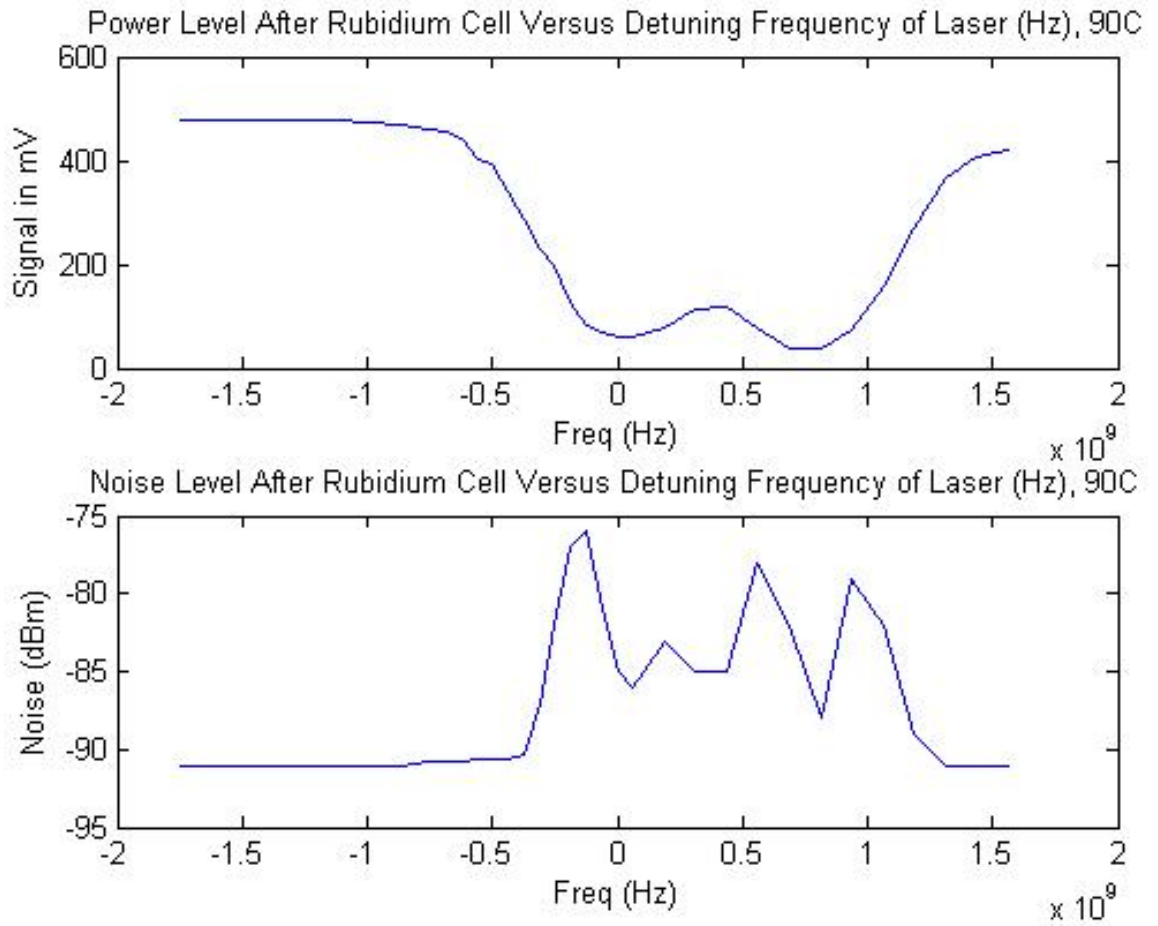


Figure 11: Power and noise of laser after propagating through a cell of ^{87}Rb at 90° . It is clear at resonance, the output noise peaks, indicating contribution from the Langevin forces

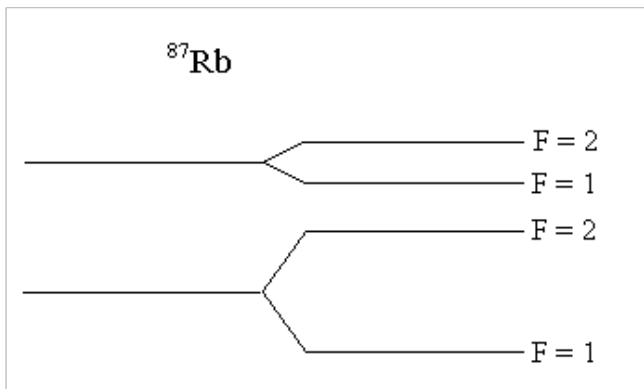


Figure 12: Energy levels of interest in the ^{87}Rb atom due to hyperfine splitting [6]

7 Methods

7.1 Introduction to Three-Level System

There are various reasons to understand the noise contribution due to the interaction between a laser and a three-level atomic system. Specifically, these interactions have practical uses, including the generation of entangled optical fields and electromagnetically induced transparency. Both of these phenomena can advance methodology in quantum computing and quantum cryptography. We employ a Rubidium vapor, in this case the Rb-87 (^{87}Rb) isotope, as our atomic system. Due to hyperfine splitting rubidium has two ground states ($F = 1,2$) and two excited states ($F' = 1,2$). Hyperfine splitting is the separation of the atomic energy levels due to the interaction between the atom's total nuclear angular momentum, I , and the total angular momentum produced by the electron, J magnetic dipole [6]. See Fig. 12 for a diagram of this splitting.

We can model the transitions between the states in several different ways. For example, the transition $2 \rightarrow 1'$ and $2 \rightarrow 2'$ can be treated as two separate two-level systems. This transition is observed experimentally in Section 6 to demonstrate the necessity of the Langevin forces in theoretical calculations, and compare experimental to theoretical trends. Additionally, the transition $2 \rightarrow 1'$ and $1 \rightarrow 1'$ can be treated as a three-level system, where $1'$ is the excited state and $1,2$ are two distinct ground states. Current experimental work in the Atomic, Molecular, and Optical group at William and Mary drives our investment in understanding the three-level transition and resulting noise. To avoid confusion, in the following sections, we denote $1'$ as a , the excited state. Likewise, 2 will be named c , one ground state, and 1 will be named b , the other ground state. See Fig. 13 for a diagram of this three-level system, the focus of the remainder of this study.

Here, the two optical fields, E_c , the pump, and E_b , the probe will have the relationship

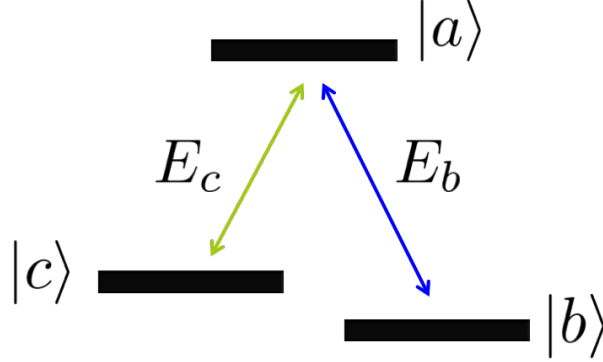


Figure 13: Three-level atomic system used to model certain transitions in ^{87}Rb

$E_c \gg E_b$. If the amplitude of E_c is large enough, we can model the behavior of this field classically with the Rabi frequency, discussed in Section 7.1. By employing this technique, calculations will be simpler as we will merely treat the field E_b quantum mechanically. Thus, the transition $b \rightarrow a$ will be primary source of Langevin noise, as we will discuss later.

7.2 Beginning with Atomic Operators

The methodology for solving for the Langevin contribution of the noise of the three-level atomic system mirrors the steps we took for the two-level system. However, first several assumptions are made about the interactions and population of the different energy levels, denoted a, b, c as mentioned in Section 7.1. First, due to the constant application of both the pump E_c at Ω_c and probe E_b at Ω_b fields, the population of the ground states can be estimated as zero. Thus, the field operator $\hat{S}_{aa} \approx 0$ which describes the atomic operator $|a\rangle\langle a|$. Similarly, $\hat{S}_{c,c} \approx 0$. In the same vein, the population of the excited state, $\hat{S}_{b,b} \approx 1$. To begin the calculations, it is necessary to have the equations of motion of the atomic field operators on hand. From here on, we use the following general forms [1]

$$\dot{S}_{c,c} = \gamma_{0r}(S_{c,c} - S_{b,b}) + \gamma_r S_{a,a} - i(\Omega_c S_{a,c} - c.c.) \quad (40)$$

$$\dot{S}_{b,b} = \gamma_{0r}(S_{c,c} - S_{b,b}) + \gamma_r S_{a,a} - i(\Omega_b S_{a,c} - c.c.) \quad (41)$$

$$\dot{S}_{a,b/c} = -\Gamma_{a,b/c} S_{b,c} - i\Omega_{b/c} * (S_{b/c,b/c} - S_{a,a}) - i\Omega_{c/b} * S_{c,a} \quad (42)$$

$$\dot{S}_{a,b/c} = -\Gamma_{a,b/c} S_{b,c} - i\Omega_{b/c} * (S_{b/c,b/c} - S_{a,a}) - i\Omega_{c/b} * S_{c,a} \quad (43)$$

7.3 Substituting into Propagation Equation

In this case, we will begin with the differential equation that describes the propagation and temporal evolution of the quantum field operators in the three-level case. By analyzing this

equation, it will be possible to specify with diffusion coefficients are necessary to calculate to eventually solve the system.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\hat{a} = igN\hat{S}_{a,b} \quad (44)$$

We chose the atomic operator $\hat{S}_{a,b}$ on the right hand side because the interaction between the states c, b will not have quantum mechanical effects such as spontaneous emission. Thus, we do not include the operator $\hat{S}_{c,b}$ directly when analyzing the behavior of the field. The first step is to solve for $\hat{S}_{a,b}$, the atomic variable of interest so we can proceed to understand the evolution of the field operator. Similar to the two-level system, we undergo a Fourier transform and find in the frequency domain

$$-i\omega\hat{S}_{a,b} = -\Gamma_{a,b}\hat{S}_{a,b} - ig\hat{a} - i\Omega_c^*\hat{S}_{c,b} + F_{a,b}(\omega) \quad (45)$$

Where any $\Gamma_{i,j}$ is a constant of the atomic medium. It is related to the linewidth of the optical transition $|i\rangle \rightarrow |j\rangle$, Zeeman splitting, and Stark shifts of the ground state [1]. We proceed to solve for $\hat{S}_{a,b}$ in terms of the Langevin forces by using the atomic equations of motion and find that

$$\hat{S}_{a,b} = \frac{-\Omega_b^* - \frac{\Omega_c^*F_{c,b}}{\Gamma_{c,b}} + F_{a,b}}{\Gamma_{a,b} + \frac{\Omega_c^*\Omega_c}{\Gamma_{c,b}} - i\omega} \quad (46)$$

This value of $\hat{S}_{a,b}$ incorporates two main concepts. First, it is expressed largely in terms of atomic constants, which can be measured. Moreover, it includes the contributions of Langevin forces in the terms $F_{c,b}$ and $F_{a,b}$, values that we will solve for in Section 8. It is important to note that complications within this schematic versus the two level system can be seen in the existence of an additional Langevin force term, $F_{c,b}$, which is related to the multiple ground states.

Now, with a value of $\hat{S}_{a,b}$ expressed in terms of the Langevin forces, we can return to the propagation equation and find the behavior of the field. Again, we mirror steps made to solve this value in the two level system. Primarily, the methodology involves moving to the frequency domain and solving the following differential equation

$$\frac{\partial}{\partial z}\hat{a} = \hat{a}\left(\frac{-g^2N}{Dc} + \frac{i\omega}{c}\right) + \frac{igN\Omega_c^*}{\Gamma_{c,b}Dc}F_{c,b} + \frac{igN}{Dc}F_{a,b} \quad (47)$$

Here, D is the denominator of Eq. 46, a constant we substituted for ease of calculation. The closed solution of this differential equation is

$$\hat{a}(z) = \hat{a}(0)e^{z(-k+r)} + \frac{f}{k+r}(e^{z(-k+r)} - 1) \quad (48)$$

where $k = \frac{-g^2N}{Dc}$, $r = \frac{i\omega}{c}$, and $f = \frac{igN\Omega_c^*}{\Gamma_{c,b}Dc}F_{c,b} + \frac{igN}{Dc}F_{a,b}$. This solution for the field is particularly promising because it assumes an identical form as that of the two level system. The first term, $\hat{a}(0)e^{z(-k+r)}$ shows the decay of the field over space without a restorative force. However, the second term, $\frac{f}{k+r}(e^{z(-k+r)} - 1)$ represents the Langevin force, initially zero. With this solution, we can continue the process to find the measurable noise of a three level system.

7.4 Solving for the Noise

After defining the behavior of \hat{a} , it is possible to calculate the measurable noise for this three-level system.

$$\langle \delta x(\omega) \delta x^\dagger(\omega) \rangle = \langle (\delta \hat{a}(\omega) + \delta \hat{a}^\dagger(\omega)) (\delta \hat{a}^\dagger(\omega) + \delta \hat{a}(\omega)) \rangle \quad (49)$$

By substituting our solution for \hat{a} , we find that

$$\begin{aligned} \langle \delta x(\omega) \delta x^\dagger(\omega) \rangle = & e^{-z((k+r)+(k+r)^*)} + \lambda^2 \left(\frac{(\Omega_c)^2}{(\Gamma_{c,b})^2} \langle F_{c,b}(\omega) F_{c,b}^\dagger(\omega) \rangle + \right. \\ & \left. \frac{(\Omega_c)^*}{(\Gamma_{c,b})} \langle F_{c,b}(\omega) F_{a,b}^\dagger(\omega) \rangle \right) + \frac{\Omega_c}{\Gamma_{c,b}} \langle F_{c,b}^\dagger F_{a,b} \rangle + \langle F_{a,b} F_{a,b}^\dagger + c.c \rangle \end{aligned} \quad (50)$$

where $\lambda = \frac{igN}{Dc(k+r)} (e^{z(-k+r)} - 1)$.

From this equation, we can see that the relevant Langevin forces in this calculation will be $\langle F_{c,b} F_{c,b}^\dagger \rangle$, $\langle F_{c,b} F_{a,b}^\dagger \rangle$, $\langle F_{a,b} F_{c,b}^\dagger \rangle$, and $\langle F_{a,b} F_{a,b}^\dagger \rangle$, and the complex conjugate of each. It will be important to calculate these eight terms, because they govern the effect of the Langevin forces in our schematic.

8 Results for Three Level System: Calculation of Noise Spectra

Calculations of certain diffusion coefficients can allow us to express the value of these Langevin forces, as seen in Eq. 16. For example, we can calculate $D_{(a,b),(a,b)} = \frac{d}{dt} \langle S_{a,b}^\dagger S_{a,b} \rangle - \langle \dot{S}_{a,b}^\dagger S_{a,b} \rangle - \langle S_{a,b}^\dagger \dot{S}_{a,b} \rangle$. Using the relationships between mean values, such as $|b\rangle\langle a||a\rangle\langle b| = 1$ and the atomic equations of motion, we find that

$$2D_{(a,b),(b,a)} = -i\Omega_b (\langle S_{a,b} \rangle - \langle S_{b,a} \rangle) \approx 0 \quad (51)$$

For the remaining Langevin forces and diffusion coefficients, $\langle F_{c,b} F_{c,b}^\dagger \rangle$, $\langle F_{c,b} F_{a,b}^\dagger \rangle$, $\langle F_{a,b} F_{c,b}^\dagger \rangle$, we aim to use the literature, as other groups have calculated these values [7]. Referencing this work, we have that

$$2D_{(c,b),(b,a)} = (\langle S_{a,c} \rangle) (\gamma_{bc}) \quad (52)$$

$$2D_{(c,b),(b,c)} = (\langle S_{a,a} \rangle) (\Gamma_{ac}) \quad (53)$$

$$2D_{(a,b),(b,c)} = (\langle S_{c,a} \rangle) (\gamma_{bc}) \quad (54)$$

And the complex conjugate terms

$$2D_{(b,c),(c,b)} = 2(\langle S_{b,b} \rangle) (\gamma_{bc}) + (\langle S_{aa} \rangle) (\Gamma_{ab}) \quad (55)$$

$$2D_{(b,c),(a,b)} = 0 \quad (56)$$

$$2D_{(b,a),(c,b)} = 0 \quad (57)$$

$$2D_{(b,a),(a,b)} = (\langle S_{b,b} \rangle)(\Gamma_a) + (\langle S_{aa} \rangle)(\Gamma_{ab})(\gamma_{bc}) \quad (58)$$

We the diffusion coefficients specified in terms of constants, the equation for the measurable noise of this three-level system simplifies to

$$\begin{aligned} \langle \delta x(\omega) \delta x^\dagger(\omega) \rangle = & e^{-z((k+r)+(k+r)^*)} + \lambda^2 \left(\frac{(\Omega_c)^2}{(\Gamma_{c,b})^2} \langle F_{c,b}(\omega) F_{c,b}^\dagger(\omega) \rangle + \right. \\ & \left. \frac{(\Omega_c)^*}{(\Gamma_{c,b})} \langle F_{c,b}(\omega) F_{a,b}^\dagger(\omega) \rangle \right) + \frac{\Omega_c}{\Gamma_{c,b}} \langle F_{c,b}^\dagger F_{a,b} \rangle + \langle F_{a,b} F_{a,b}^\dagger \rangle + c.c \end{aligned} \quad (59)$$

As we did for the two-level system, we would like to plot the behavior of this relationship as a function of frequency at various densities. To do this, we employ a Mathematica package, the Atomic Density Matrix Package, that handles atomic work. With this tool, we are able to calculate the atomic operators. Since the diffusion coefficients and Langevin forces can be written in terms of these operators, the package will be particularly useful.

The Atomic Density Matrix Package was written by Dr. Simon Rochester from Rochester Scientific to model atomic systems while accounting for various effects, such as EIT, Zeeman structures, hyperfine splitting, and the Stark effect. For this project, we use documentation for a three-level EIT system. We define a system with two ground states and one excited state, identical to our schematic of interest. Then, we can specify variable optical fields between the $|b\rangle \rightarrow |a\rangle$ and $|c\rangle \rightarrow |a\rangle$ states. The Atomic Density Matrix package has the ability to identify the Hamiltonian for the system based on the previous conditions given (such as external electric and magnetic fields applied on the system). Furthermore, we can apply the rotating wave approximation to exclude Hamiltonian terms with high frequencies of the form $\Omega_f + \omega_0$ where Ω_f is that of an applied field and ω_0 is that of the atomic transition. Next, we add constants such as γ which repopulate the atomic matrices after unrealistic losses. Then, the program asserts the Liouville equation, which governs the evolution of the density matrix over time, and solves the resulting system of equations to find the values of the atomic density operators.

Next, we aim to calculate and plot the trends of the three-level system noise function we obtained theoretically. Since we can express our diffusion coefficients in terms of the atomic operators, the Mathematica calculations above for the atomic operators will prove especially useful.

9 Future Work

First, we would like to be able to model and plot the noise function, Eq. 59, for a three-level system as a sum of the field term and Langevin force term. Then, would like to verify the behavior experimentally at various frequencies and densities, employing the ^{87}Rb transitions $2 \rightarrow 1'$ and $1 \rightarrow 1'$ as discussed in Section 7.1. Once the noise functions are verified, we will proceed to more complex experiments. An immediate goal is to understand the FWM experiment, where a three-level optical system is pumped with frequency Ω , and two others correlated beams are generated from this process. Moreover, other experiments include passing squeezed light through an atomic ensemble under EIT conditions. In the past, the output noise was higher than expected which interfered with measurements. By modeling the process theoretically, we are able to understand a cohort of experiments with more depth.

10 Conclusion

We have demonstrated the necessity of incorporating the Langevin formalism to approach the interaction between light and an atomic system beyond a noiseless approximation. By passing a field of variable frequency through a heated cell of ^{87}Rb atoms, we find that the noise, or fluctuations in the original beam, increases at high atomic density and resonance. At resonance, particles move to the excited state, where they may be spontaneously emitted due to interactions between the atom and the vacuum field (a purely quantum mechanical effect). These particles that are emitted lead to two main effects in the behavior of the original field at specific absorption frequencies and high densities: (1) the mean value of the beam decreases (2) the noise of the beam increases. We can apply a formalism developed for classical mechanics to solve Brownian motion; where the average momentum particle traveling through a fluid decreases over space in time, but interactions with the fluid causes fluctuations in the momentum.

To properly treat the methodology of applying the Langevin forces into a three-level Λ system, we begin with a simple model of a two-level, semiclassical system. However, we cannot fully account for the known effect of spontaneous emission, and move to a quantum mechanical treatment of the applied field. In doing so, we find that we can make an analogy between the behavior of the field due to spontaneous emission and Brownian motion. When applying the Langevin formalism to an atomic system, we begin with a field and atomic reservoir. By exploring the methodology used for this simple schematic, we proceed to more complex systems. Then, with the equations of motion for atomic variables and propagation equation, we solve for the noise of a two-level system. We plot this noise, and find that the general form of the functions matches those that we achieved experimentally, testing two transitions

of ^{87}Rb . Those general forms are characterized by an increase in the Langevin force noise at resonance at high atomic density, and an increase in the field contribution further from this frequency, or at low densities. We apply an identical treatment to the three-level system, and plan to graph the noise functions in a similar way with the aid of Mathematica and the Atomic Density Matrix package. In the future, these noise functions will help members of the AMO group at William and Mary analyze experiments under four-wave mixing and EIT conditions; and why the noise of a given output field is higher than expected.

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