Squeezed states

$$\hat{E}_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(\hat{a}\, e^{ikz-i\omega t} + \hat{a}^+ e^{-ikz+i\omega t}\right) =$$

$$= 2\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left[\hat{X}_1 \cos(kz-i\omega t) + \hat{X}_2 \sin(kz-i\omega t)\right]$$

$\hat{X}_1, \hat{X}_2$ – quadrature operators

$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^+)$ $\qquad$ $\hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^+)$

intensity quadrature $\qquad$ phase quadrature

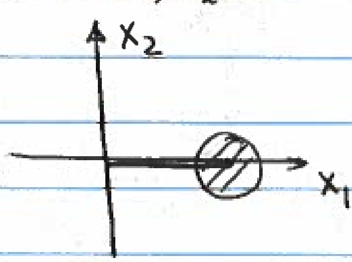
In general, $\hat{X}_\chi = \frac{1}{2}(\hat{a}e^{-i\chi} + \hat{a}^+ e^{i\chi})$

$\chi = 0 \Rightarrow \hat{X}_1$ $\qquad$ $\chi = \pi/2 \Rightarrow \hat{X}_2$

Next week, we will show that $\langle X_\chi \rangle$ and $\Delta X_\chi = \sqrt{\langle \hat{X}_\chi^2 \rangle - \langle \hat{X}_\chi \rangle^2}$ can be measured experimentally.

Two orthogonal quadratures do not commute

$[\hat{X}_1, \hat{X}_2] = i/2$ $\qquad$ $\Delta X_1^2\, \Delta X_2^2 \geq \frac{1}{16}$ or $\Delta X_1 \cdot \Delta X_2 \geq \frac{1}{4}$



Example: $\hat{X}_1$ – intensity quadrature, what one would measure with no extra equipment

Average Photocurrent $\propto \langle \hat{X}_1 \rangle^2$

Photocurrent fluctuations $\propto \Delta X_1$

For more precise measurements we want less noise $\Rightarrow$ reduced ΔX_1 (or for more sophisticated measurement shemes – ΔX_χ)

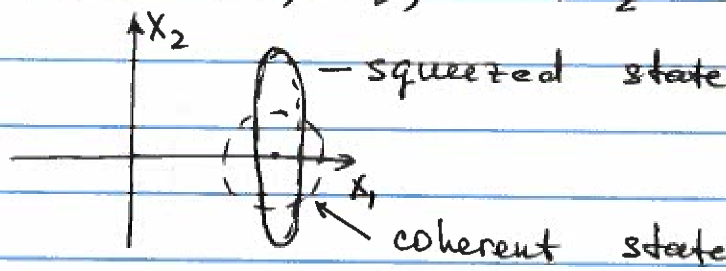
Coherent state - minimum uncertainty state $\langle \hat{X}_1 \rangle_\alpha = \langle \alpha | \hat{X}_1 | \alpha \rangle = \mathrm{Re}\,\alpha$

$$\langle \hat{X}_2 \rangle_\alpha = \langle \alpha | \hat{X}_2 | \alpha \rangle = \mathrm{Im}\,\alpha$$

$$\Delta X_1 = \Delta X_2 = \Delta X_\chi = 1/2$$

determines a shot noise limit of optical measurements

But what if one needs / wants to reduce the fluctuations below this limit? It is possible in, e.g., $\Delta X_1 < \frac{1}{2}$ but $\Delta X_2 > \frac{1}{2}$



Squeezing operator $\hat{S}(\xi) = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}$

Squeezing parameter: $\xi = r e^{i\theta}$ $\quad \hat{S}^+(\xi) = \hat{S}(-\xi) = e^{-\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{+2})}$

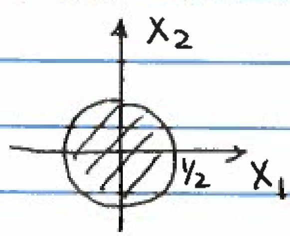
Baker - Hausdorf lemma

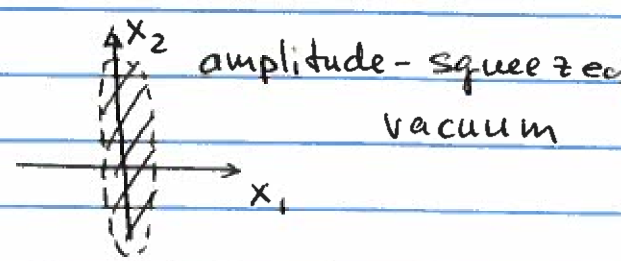
$$\hat{S}^+(\xi)\, \hat{a}\, \hat{S}(\xi) = \hat{a} \cosh r - \hat{a}^+ e^{i\theta} \sinh r$$

$$\hat{S}^+(\xi)\, \hat{a}^+ \hat{S}(\xi) = \hat{a}^+ \cosh r - \hat{a}\, e^{-i\theta} \sinh r$$

Squeezed vacuum state $|\xi\rangle = \hat{S}(\xi)|0\rangle$

Coherent vacuum





Zero average electric field

$$\langle \xi | \hat{a} | \xi \rangle = \langle 0 | \hat{S}^{+}(\xi) \hat{a} \, \hat{S}(\xi) | 0 \rangle = 0$$

$$\langle \xi | \hat{X}_{1,2} | \xi \rangle = 0$$

$$\langle \xi | \hat{a}^2 | \xi \rangle = \langle 0 | \hat{S}^{+} \hat{a}^2 \hat{S} | 0 \rangle = \langle 0 | \hat{S}^{+} \hat{a} \hat{S} \hat{S}^{+} \hat{a} \hat{S} | 0 \rangle$$

using that, we can calculate the quadrature variance

$$(\Delta X_{1,2})^2 = \frac{1}{4} \left(\cosh^2 r + \sinh^2 r \mp 2 \sinh r \cosh r \cos\theta \right) =$$
$$= \frac{1}{8} \left((e^{2r} + e^{-2r}) \mp (e^{2r} - e^{-2r}) \cos\theta \right)$$

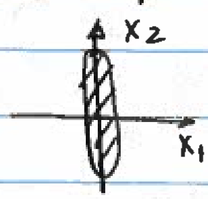
$\theta = 0$ intensity squeezing

$$\Delta X_1^2 = \frac{1}{4} e^{-2r}$$
$$\Delta X_2^2 = \frac{1}{4} e^{2r}$$

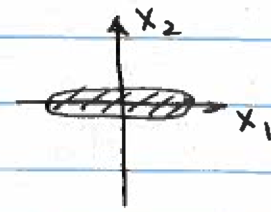
$\theta = \frac{\pi}{2}$ phase squeezing

$$\Delta X_1^2 = \frac{1}{4} e^{2r}$$
$$\Delta X_2^2 = \frac{1}{4} e^{-2r}$$

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

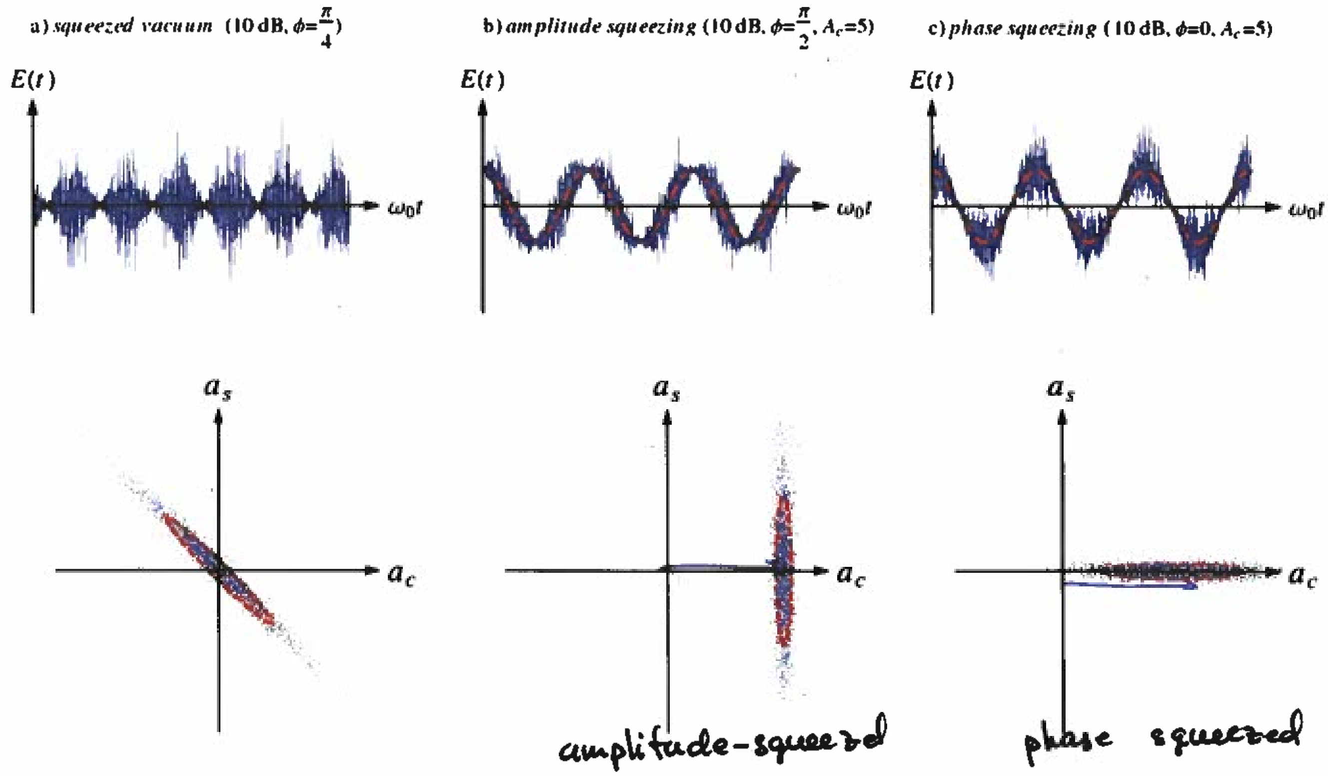


still min uncertainty states



Typically for practical purposes the amount of squeezing is characterized by the ratio the ~~noise~~ fluctuations ~~is~~ are reduced below the shot noise (coherent state limit) since it removes the need for calibration

$$\frac{(\Delta E_{sq})^2}{(\Delta E_{coh})^2} = \frac{(\Delta X_{sq})^2}{(\Delta X_{coh})^2} = \frac{\frac{1}{4} e^{-2r}}{\frac{1}{4}} \to e^{-2r}$$


a) squeezed vacuum (10 dB, $\phi=\frac{\pi}{4}$)
b) amplitude squeezing (10 dB, $\phi=\frac{\pi}{2}$, A_c=5)
c) phase squeezing (10 dB, ϕ=0, A_c=5)
$E(t)$
$\omega_0 t$
a_s
a_c
amplitude-squeezed
phase squeezed

Does squeezed vacuum has energy?

$$\hat{H}_{EM} = \hbar\omega\,(\hat{n}+\tfrac{1}{2})$$

Coherent vacuum $\langle E_{vac}\rangle = \langle 0|\hat{H}_{EM}|0\rangle =$

$= \hbar\omega\,\langle 0|\hat{n}+\tfrac{1}{2}|0\rangle = \tfrac{1}{2}\hbar\omega$ (zero-point energy)

Squeezed vacuum

$$\langle E_{SV}\rangle = \langle\xi|\hat{H}_{EM}|\xi\rangle = \hbar\omega\,\langle 0|\hat{S}^{+}\hat{a}^{+}\hat{a}\hat{S}|0\rangle + \tfrac{1}{2}\hbar\omega$$

$$\langle 0|\hat{S}^{+}\hat{a}^{+}\hat{a}\hat{S}|0\rangle = \langle 0|\hat{S}^{+}\hat{a}^{+}\hat{S}\,\hat{S}^{+}\hat{a}\hat{S}|0\rangle =$$

$$= \langle 0|(\hat{a}^{+}\cosh r - \hat{a}\,e^{-i\theta}\sinh r)(\hat{a}\cosh r - \hat{a}^{+}e^{i\theta}\sinh r)|0\rangle$$

$$= \ldots\,\langle 0|\hat{a}^{+2}|0\rangle + \ldots\,\langle 0|\hat{a}^{2}|0\rangle + \ldots\,\langle 0|\hat{a}^{+}\hat{a}|0\rangle +$$

$$+ \sinh^2 r\,\langle 0|\underbrace{\hat{a}\hat{a}^{+}}_{(\hat{a}^{+}\hat{a}+1)}|0\rangle = \sinh^2 r$$

$$\langle E_{SV}\rangle = \hbar\omega\,(\sinh^2 r + \tfrac{1}{2})$$

$$\langle n\rangle_{SV} = \sinh^2 r > 0$$

Squeezed vacuum actually has some photons in it (and extra energy)

Squeezed vacuum – photon statistics

$$p_n = |\langle n|\xi\rangle|^2 = |\langle n|\hat{S}(\xi)|0\rangle|^2 =$$

$$= |\langle n|\, e^{\frac{1}{2}(\xi^*\hat{a}^{+} - \xi\hat{a}^{+2})}\,|0\rangle|^2 =$$

$$= |\langle n| \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{(\xi^*\hat{a}^2 - \xi\hat{a}^{+2})^k}{k!} |0\rangle|^2$$

Remarkably, $p_{2m+1} = 0$

Only even number of photons are present (in the case of an ideal squeezed vacuum state)

$$p_{2m} = \binom{2m}{m} \frac{1}{\cosh r} \left(\frac{1}{2}\tanh r\right)^{2m} \qquad (\text{for } \theta = 0)$$

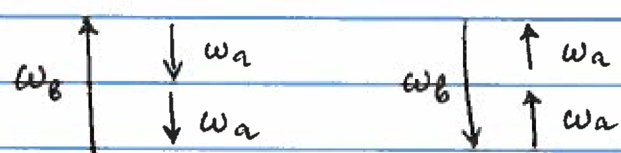
We may have suspected that considering the shape of the squeezing operator

$$|\xi\rangle = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{+2})}\;|0\rangle$$

↑ result of the squeezing nonlinear interaction

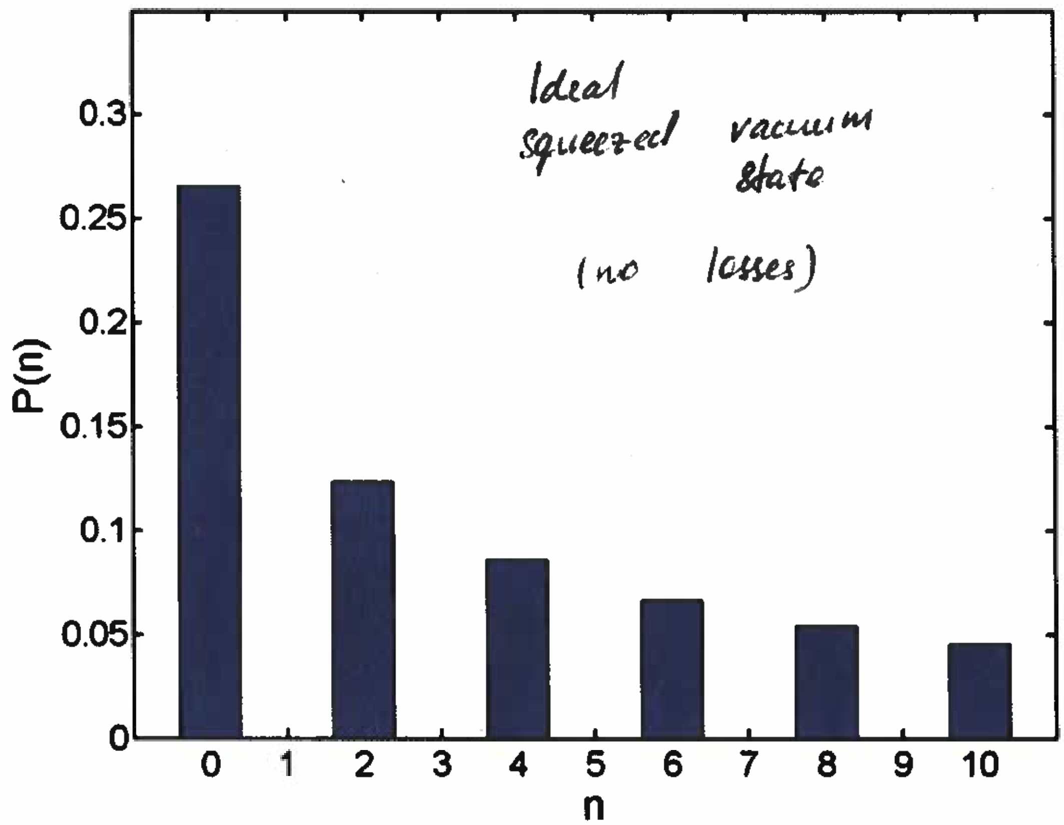
$$\hat{H} \propto \beta^* \hat{a}^2 - \beta\hat{a}^{+2}$$

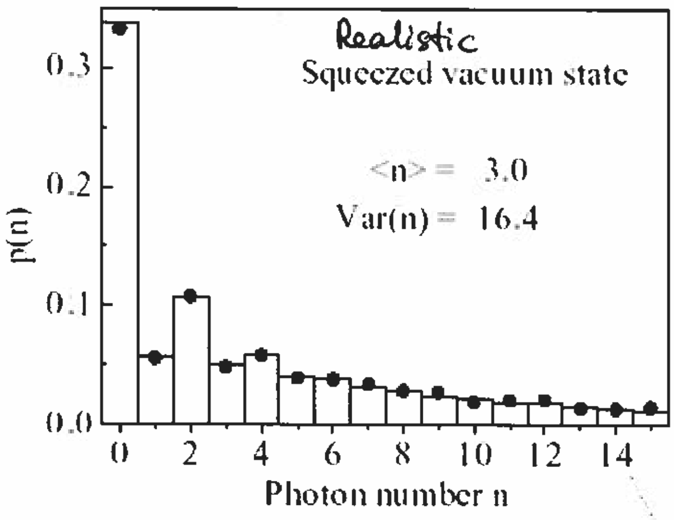
parametric down conversion



$\hat{b}^* \hat{a}^{+2}$ $\qquad$ $\hat{b}^+ \hat{a}^2$

$$\hat{H}_{PDC} \propto \hat{b}\,\hat{a}^{+2} - \hat{b}^{+}\hat{a}^{2}$$


Ideal
squeezed vacuum
state
(no losses)
0.3
0.25
0.2
0.15
0.1
0.05
0
P(n)
0 1 2 3 4 5 6 7 8 9 10
n



Any loss of photons breaks the symmetry, introducing photons into odd-number state, thus corrupting squeezing

Squeezed coherent state

Coherent state $\to$ displaced coh. vacuum

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$$

Squeezed coh. state $\to$ displaced squeezed vacuum

$$|\alpha,\xi\rangle = \hat{D}(\alpha)|\xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$$

average photon number

$$\langle n\rangle = |\alpha|^2 + \sinh^2 r$$

$$\langle\alpha,\xi|\Delta\hat{X}_i|\alpha,\xi\rangle = \langle\xi|\Delta\hat{X}_i|\xi\rangle = \frac{1}{2}e^{\pm ir}$$

depending on θ

Two-mode squeezing : consider two modes of an EM field $\omega_1, \omega_2 \to \hat{a}_1, \hat{a}_2$

$$\hat{S}_{TMS} = e^{\frac{1}{2}(\xi\hat{a}_1^+\hat{a}_2^+ - \xi^*\hat{a}_1\hat{a}_2)}$$

$$|TMSV\rangle = \hat{S}_{TMS}|0,0\rangle = \frac{1}{\cosh r}\sum_{n=0}^{\infty}(e^{-2i\theta}\tanh r)^n|n,n\rangle$$

The number of photons in two states is perfectly correlated, even though each of individual fields will have thermal statistics with the Boltzman distribution $e^{-\hbar\omega/k_BT} = \tanh r \quad (\theta=0)$

Squeezed coherent state

$$|\alpha,\xi\rangle = \hat{S}(\xi)|\alpha\rangle$$

$$\langle\alpha,\xi|\hat{X}_{1,2}|\alpha,\xi\rangle = \langle\alpha|\hat{S}^{\dagger}(\xi)\hat{X}_{1,2}\hat{S}(\xi)|\alpha\rangle =$$

$$= \frac{1}{2}e^{\pm r}(\alpha e^{i\theta} \pm \alpha^* e^{-i\theta})\binom{1}{i}$$

$$E = \frac{E}{r} e^{-(\gamma+i\omega)(t-r/c)} \theta(t-r/c)$$

$$|E|^2 = \frac{E}{r}\int_{r/c}^{\infty} e^{-2\gamma(t-r/c)} \cancel{\theta^2(t-\frac{r}{c})} =$$

$$= \frac{|E|^2}{r^2} \cancel{e^{2\gamma r/c}} \frac{1}{2\gamma} \cancel{e^{-2\gamma r/c}} =$$

$$= \frac{|E|^2}{2\gamma r^2}$$

$$E(t)E^{\dagger}(\tau) = \frac{|E|^2}{r^2}\int_{-\infty}^{+\infty} \cancel{e^{+2\gamma(t-r/c)}\,\theta(t-\frac{r}{c})}$$

$t > r/c$

$t > r/c - \theta$

$$\int_{-r/c}^{+\infty} e^{-(\gamma+i\omega)(t-r/c)} e^{-(\gamma-i\omega)(t+\tau-r/c)}$$

$$\cancel{\theta(t-\frac{r}{c})\theta(t+\tau-\frac{r}{c})}$$

$t - |\tau| - \frac{r}{c}$

$$= \cancel{e^{+2\gamma r/c}} e^{-(\gamma-i\omega)\tau} \int_{r/c+|\tau|}^{\infty} e^{-2\gamma t}\, dt = \frac{e}{2\gamma} \cancel{e^{-2\gamma(r/c+|\tau|)}}$$

$$E(t)E(t+\tau) = e^{-(\gamma-i\omega)|\tau|} \frac{|E|^2}{r^2 2\gamma}$$

$$g^{(1)} = e^{-(\gamma-i\omega)\tau}$$