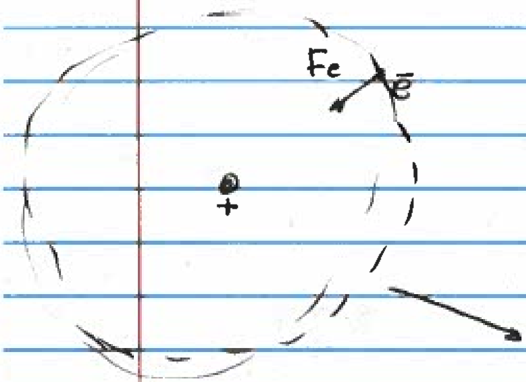
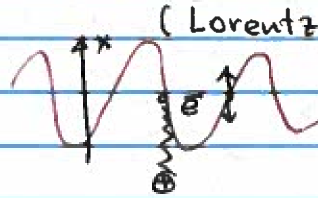
# Atom in an electromagnetic field: classical story




Some necessary assumptions

1. EM wave is <u>weak</u> compare to coulumb interaction (weak perturbation)

~~oscillator~~ atom = oscillator (Lorentz atom)




No em field → no extra motion
With em field → oscillations (induced dipole moment)

2. We neglect any spatial variations across electron orbit

In principle $E(z,t) = E_0 \cos\left(\frac{2\pi}{\lambda} z - \omega t + \varphi\right)$

Characteristic lengthscale of e-m wave (visible) $\sim 0.1 - 1\ \mu m$

Characteristic size of an atom $\sim 1 - 10$ Å

a few orders of magnitude smaller

In fact: estimate for the size of the atom

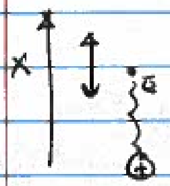
$$r \sim a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{me^2} = \frac{\hbar}{mc\alpha}$$

Characteristic frequencies $\omega \sim \frac{R_E}{\hbar} = \frac{mc^2\alpha^2}{\hbar}$

$$\lambda = \frac{2\pi c}{\omega} \approx \frac{2\pi\hbar}{mc\alpha^2}$$

$$\frac{r}{\lambda} \sim \frac{\alpha}{2\pi} \qquad \text{small!}$$

Only time-dependent EM field $E(t) = E_0 \cos\omega t$



$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{e}{m} E(t) = \frac{e}{m} E_0 \cos\omega t$$

$\omega_0$ – "natural" oscillation frequency
$\gamma$ – damping constant

$$\gamma = \frac{\omega_0^2 r_0}{3c} = \frac{\omega_0^2}{3c} \underbrace{\frac{1}{4\pi\varepsilon_0}\left(\frac{e^2}{mc^2}\right)}_{r_0 \text{ - classical electron radius}}$$

in case of radiative damping

Usually $\gamma$ is just a phenomenological constant

It is convenient to use complex exponents

$$E(t) = E^+ + E^- = \frac{1}{2}E_0\, e^{-i\omega t} + \frac{1}{2}E_0\, e^{i\omega t}$$

as long as the problem is linear, we can keep track of only the first term

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{eE_0}{2m} e^{-i\omega t}$$

$$x(t) = \frac{i}{2}\frac{eE_0}{2m\omega} e^{-i\omega t} \frac{1}{\gamma + \frac{i(\omega_0^2-\omega^2)}{2\omega}} \quad (+\ c.c.)$$

For all the systems we are going to consider $\gamma \ll \omega_0$, and thus the strongest responce is near the resonance $\omega \approx \omega_0$ (or, more exactly $|\omega_0 - \omega| < \gamma$). Then

$$\omega_0^2 - \omega_0^2 = (\omega_0+\omega)(\omega_0-\omega) \approx 2\omega(\omega_0-\omega)$$

$$x(t) \to \frac{i}{2}\frac{eE_0}{2m\omega_0}\frac{1}{\gamma + i(\omega_0-\omega)} + c.c.$$

What does it mean for the EM field?

EM waves in a dielectric medium

Maxwell's equations

$$\nabla\vec{B} = 0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{D} = \varepsilon_0 \vec{E} + \underset{\text{polarization}}{\vec{P}}$$

$$\nabla\vec{D} = 0 \qquad \nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} \qquad \vec{H} = \frac{1}{\mu_0}\vec{B}$$

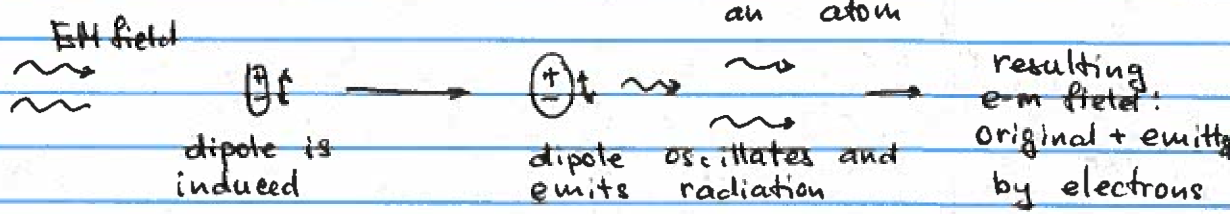
Excluding the magnetic field → wave equation $\vec{E}$

$$\underbrace{\nabla\times[\nabla\times\vec{E}]}_{} = -\frac{\partial}{\partial t}[\nabla\times\vec{B}] = -\mu_0\frac{\partial}{\partial t}[\nabla\times\vec{H}] = -\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$$

$= \nabla(\nabla\vec{E}) - \nabla^2\vec{E} \approx -\nabla^2\vec{E}$ (for the transverse EM field $\nabla\cdot\vec{E} \propto \vec{k}\cdot\vec{E}_0$

$$\underbrace{-\nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}_{\text{wave propagation}} = \underbrace{-\mu_0\frac{\partial^2\vec{P}}{\partial t^2}}_{\text{medium response (macroscopic)}}$$

$$\vec{P} = \sum_{\text{volume}} \vec{d}_i = N\langle\vec{d}\rangle = N\underbrace{\langle -e\vec{r}\rangle}_{\text{for an electron in an atom}}$$




In our 1D model

$$\vec{P}_{(z)} = \sum_{\substack{\text{unit}\\ \text{volume}(z)}} \vec{d}_i = \sum -e x(t)\Big|_z = \frac{1}{2}\frac{e^2 N}{2 i m\omega}\frac{e^{ikz-i\omega t}}{\gamma + i(\omega-\omega_0)} E_0(z,t)$$

Slowly - varying amplitudes

$$E(z,t) = \frac{1}{2} \underbrace{E_0^{(+)}(z,t)}_{\text{slowly varying}} e^{ikz - i\omega t} + c.c$$

$$P(z,t) = \frac{1}{2} \overbrace{P_0^{(+)}(z,t)} e^{ikz - i\omega t} + c.c.$$

We assume that any spatial or temporal changes in $E_0$ or $P_0$ happen much slower than the oscillations of EM field phase

$$\left|\frac{\partial E_0}{\partial t}\right| \ll \omega |E_0| \qquad \left|\frac{\partial E_0}{\partial z}\right| \ll k|E_0| \qquad \text{(same for P)}$$

Then we can simplify the wave equation

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) E_0^{(+)}(z,t) = -\frac{k}{2i\varepsilon_0} P_0^{(+)}(z,t)$$

$$E_0^{(+)}(z,t) = E_0(z,t)\, e^{i\varphi}$$

Without loss of generocity we can assume that $E_0(z,t)$ is a real function. Note, that in general $P_0(z,t)$ will still be complex, and it is convenient to split it into real and imaginary parts

Since $\vec{P} = \varepsilon_0 \chi \vec{E}$ ($\chi$ – linear susceptibility)

$$P_0(z,t) = \varepsilon_0 (\chi' + i\chi'') E_0(z,t)$$

$$\begin{cases} \dfrac{\partial E_0}{\partial z} + \dfrac{1}{c}\dfrac{\partial E_0}{\partial t} = -\dfrac{k}{2\varepsilon_0} \mathrm{Im}(P_0) = -\dfrac{k}{2}\chi'' E_0 \\[2ex] \dfrac{\partial \varphi}{\partial z} + \dfrac{1}{c}\dfrac{\partial \varphi}{\partial t} = \dfrac{k}{2}\chi' \end{cases}$$

Let's consider a special case - constant (i.e time-independent) EM field $E_0(z,t) \neq E_0(t)$

$$\frac{\partial E_0}{\partial t} = 0 \qquad \frac{\partial E_0}{\partial z} = -\frac{k}{2}\chi'' E_0$$

$$E_0(z) = E_0(0)\, e^{-k\chi''/2 \cdot z} = E_0\, e^{-\alpha z}$$

Beer's law

Absorption coefficient $\alpha = \frac{1}{2} k\chi''$

$$\frac{\partial \varphi}{\partial t} = 0 \qquad \frac{\partial \varphi}{\partial z} = \frac{k}{2}\chi' \qquad \varphi(z) = \varphi(0) + \frac{k}{2}\chi' z$$

Total phase of the propagating EM field

$$kz - \omega t + \varphi(z) = kz - \omega t + \varphi(0) + \frac{k}{2}\chi' z =$$

$$= k\left(1 + \frac{1}{2}\chi'\right)z - \omega t + \underbrace{\varphi(0)}_{=0} = \frac{2\pi}{\lambda_0}\underbrace{\left(1 + \frac{1}{2}\chi'\right)}_{\text{refractive index}} - \omega t$$

Note: all we did so far regarding the optical field will be applicable for a lot of semi-classical (quantum atom + classical EM field) ~~approx~~ model. There we will be able to calculate $\chi'$ and $\chi''$ more accurately. However, even though some important features will be possible even in the Lorentz atom model

$$\chi = \frac{N}{i\varepsilon_0}\frac{e^2}{2m\gamma\omega_0}\frac{\gamma}{\gamma + i(\omega_0 - \omega)}$$

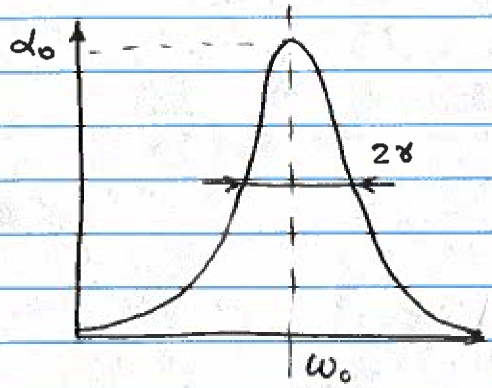
Absorption coefficient

$$\alpha(\omega) = \mathrm{Im}\left[\frac{1}{2}k\chi\right] = \mathrm{Im}\left[\frac{1}{2i}\frac{N}{\varepsilon_0}\frac{ke^2}{2m\gamma\omega_0}\frac{\gamma}{\gamma + i(\omega_0 - \omega)}\right]$$

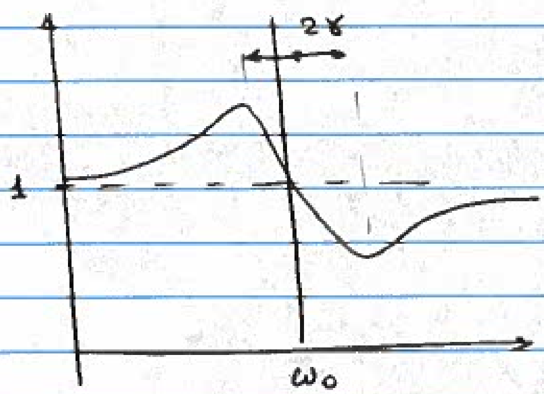
$$\alpha(\omega) = \alpha_0 \frac{\gamma^2}{\gamma^2 + (\omega - \omega_0)^2} \qquad \alpha_0 = \frac{kN}{2\varepsilon_0}\frac{e^2}{2m\gamma\omega_0}$$

Refractive index

$$n(\omega) = 1 + \frac{1}{2}\chi' = 1 + \alpha_0 \frac{\gamma(\omega_0-\omega)}{\gamma^2+(\omega_0-\omega)^2}$$




resonant absorption




resonant dispersion