Quantization of electro-magnetic field

Classical electrodynamics: we describe e-m field using electric and/or magnetic field amplitudes, guided by the Maxwell's equations:

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \nabla \times \vec{H} = +\dfrac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{D} = 0 & \nabla \cdot \vec{B} = 0 \end{cases}$$

(no sources)

$\vec{D} = \varepsilon_0 \varepsilon \vec{E}$

$\vec{B} = \mu_0 \mu \vec{H}$

Quantum electrodynamics: all fields become <u>operators</u>: $\hat{\vec{E}}, \hat{\vec{B}}, \hat{\vec{D}}, \hat{\vec{H}}$

and the electromagnetic field is now described by its quantum state $|\psi\rangle$

Thus, we now ~~can~~ have to measure the ~~values~~ average values of the field amplitudes in each state.

$$\vec{E} = \langle \psi | \hat{\vec{E}} | \psi \rangle$$

However, we still expect that Maxwell's equations are valid for <u>any</u> state of e-m field.

$$\nabla \times \langle \psi | \hat{\vec{E}} | \psi \rangle = -\frac{\partial}{\partial t} \langle \psi | \hat{\vec{B}} | \psi \rangle$$

$$\langle \psi | \nabla \times \hat{\vec{E}} + \frac{\partial \hat{\vec{B}}}{\partial t} | \psi \rangle = 0 \quad \text{for any } |\psi\rangle$$

$$\nabla \times \hat{\vec{E}} = -\frac{\partial \hat{\vec{B}}}{\partial t}$$

(same for three other equations)

Similarly, $\hat{\vec{D}} = \varepsilon_0 \varepsilon \hat{\vec{E}}$, $\hat{\vec{B}} = \mu_0 \mu \hat{\vec{H}}$

Classical e-m field energy

$$\mathcal{H} = \frac{1}{2}\int dV\,(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$$

Corresponding QED Hamiltonian

$$\hat{\mathcal{H}} = \frac{1}{2}\int dV\,(\hat{\vec{E}}\cdot\hat{\vec{D}} + \hat{\vec{B}}\cdot\hat{\vec{H}}) = \frac{1}{2}\int dV\,(\varepsilon_0\varepsilon\,\hat{E}^2 + \frac{1}{\mu_0\mu}\hat{B}^2)$$

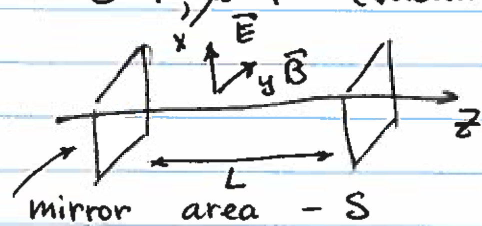
$\hat{\vec{E}}$ and $\hat{\vec{B}}$ are not independent but connected through Maxwell's equations.

This Hamiltonian is very similar to SHO Hamiltonian

We will consider the simplest configuration and then generalize

Plane E-M field b/w two perfect mirrors

$\varepsilon = 1$, $\mu = 1$ (vacuum), $E(z=0) = E(z=L) = 0$



Classical solution

$$\nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

$\vec{E} = E\vec{e}_x$, $\vec{B} = B\vec{e}_y$

Spatial modes that automatically satisfy the boundary conditions $E_x \propto \sin\left(\frac{\pi n z}{L}\right)$ $n = 1, 2, \ldots$

For each spatial mode (i.e. unique standing wave)

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\,q(t)\,\sin kz \qquad k = \frac{\pi n}{L} = \frac{\omega}{c}$$

$V = S\cdot L$ volume

At this moment we "guessed" normalization, but we could leave it as unknown constant and then normalize later.

$$\vec{E}_x = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q(t) \sin kz \cdot \vec{e}_x$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \frac{\partial B_y}{\partial z} = \frac{1}{c^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, \dot{q} \sin kz$$

$$B_y = -\underbrace{\frac{1}{c^2 k}}_{= c\cdot\omega}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, \dot{q}(t) \cos kz = -\sqrt{\frac{2}{V\varepsilon_0 c^2}}\, \dot{q}(t)\cos kz$$

Energy (hamiltonian)

$$\overset{\text{4}}{H} = \frac{1}{2}\int_V dV \left(\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2\right) =$$

(back to normal $\hat{H}$, since we don't use magnetic H)

$$= \frac{1}{2}\int_V dV \left(\cancel{\varepsilon_0} \cdot \frac{2\omega^2}{V\cancel{\varepsilon_0}} q^2 \sin^2 kz + \frac{1}{\mu_0}\frac{2}{V\varepsilon_0 c^2}\dot{q}^2 \times \cos^2 kz\right)$$

$$\int_V \sin^2 kz \, dV = S\int_0^L \sin^2 kz \, dz = S\cdot\frac{L}{2} = V/2$$

same for $\int_V \cos^2 kz \, dV = V/2$

$$H = \frac{1}{2}\cdot\frac{V}{2}\left(\frac{2\omega^2}{V} q^2 + \frac{1}{\cancel{\mu_0\varepsilon_0}}\frac{2}{V\cdot \cancel{c^2}}\dot{q}^2\right) =$$

$$= \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 \qquad \text{SHO}$$

Quantum analog: $\hat{H} = \frac{1}{2}\hat{\dot{q}}^2 + \frac{1}{2}\omega^2\hat{q}^2 = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$

$$[\hat{q},\hat{p}] = i\hbar$$

Annihilation and creation operators (ladder operators)

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

$$[\hat{a},\hat{a}^+] = 1 \qquad \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H},\hat{a}] = i\omega[\hat{a}^+\hat{a},\hat{a}] = i\omega(\hat{a}^+\hat{a}\hat{a} - \overbrace{\hat{a}\hat{a}^+}^{1+\hat{a}^+\hat{a}}\hat{a}) = -i\omega\hat{a}$$

$$\hat{a}(t) = e^{-i\omega t}\hat{a}, \quad \hat{a}^+(t) = e^{i\omega t}\hat{a}^+$$

Plane EM wave inside the cavity

$$\hat{E}_x = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right) \sin kz$$

$$\hat{B}_y = \frac{1}{c}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} - \hat{a}^+ e^{i\omega t}\right) \cos kz$$

Now we need to describe the <u>state</u> of electromagnetic field

Number (or Fock) states

$$\hat{n} = \hat{a}^+\hat{a} \qquad \hat{n}|n\rangle = n|n\rangle$$

$|n\rangle$ – eigenstate of the number operator, describing the state with known photon number.

$|0\rangle$ – vacuum state (no photons)

$|1\rangle$ – single-photon state

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^+a + \tfrac{1}{2}\right)|n\rangle = \hbar\omega\left(\hat{n} + \tfrac{1}{2}\right)|n\rangle =$$

$$= \underbrace{\left(n+\tfrac{1}{2}\right)\hbar\omega}_{\text{energy } E_n \text{ of the state } |n\rangle} \cdot |n\rangle$$

$$E_n = \underbrace{n\hbar\omega}_{\hbar\omega \times \#\text{ of quanta}} + \frac{1}{2}\hbar\omega$$

vacuum fluctuations

Why $\hat{a}$ is called annihilation operator?

$$\hat{n}\left(\hat{a}|n\rangle\right) = \hat{a}^+\hat{a}\left(\hat{a}|n\rangle\right) = \left(-1 + \hat{a}\hat{a}^+\right)\left(\hat{a}|n\rangle\right) =$$

$$= \hat{a}\underbrace{\hat{a}^+\hat{a}}_{\hat{n}}|n\rangle - \hat{a}|n\rangle = n\,\hat{a}|n\rangle - \hat{a}|n\rangle = (n-1)\,\hat{a}|n\rangle$$

$\hat{n}\left(\hat{a}|n\rangle\right) = (n-1)\left(\hat{a}|n\rangle\right) \Rightarrow \hat{a}|n\rangle$ is the eigenstate of $\hat{n}$ with eigenvalue $(n-1)$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

and $$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{E}_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^+ e^{i\omega t} \right) \sin kz$$

Measured mean value of the electric field

$$\langle n | \hat{E} | n \rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \Big(\underbrace{\langle n | \hat{a} | n \rangle}_{=0} e^{-i\omega t} + \underbrace{\langle n | \hat{a}^+ | n \rangle}_{=0} e^{i\omega t} \Big) \times \sin kz$$

$\langle n | \hat{E} | n \rangle = 0$ no defined electric field

The number state is highly non-classical, i.e. they do not have classical analoges

Fluctuations of electro-magnetic field

$$\Delta E_x = \sqrt{\langle \hat{E}_x^2 \rangle - \underbrace{\cancel{\langle E_x \rangle^2}}_{=0}} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \sqrt{\langle n | | \hat{a} e^{-i\omega t} + \hat{a}^+ e^{i\omega t} |^2 | n \rangle} \times |\sin kz|$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\sin kz| \underbrace{\sqrt{\langle n | \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} | n \rangle}}_{\sqrt{2n+1}}$$

Note that even for $n=0$ (vacuum state)

$$\Delta E_x \neq 0 = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\sin kz|$$

In quantum mechanics vacuum is an active participant, not empty nothingness.

Vacuum fluctuations

Each mode (possible spatial configuration) contain vacuum energy, even if there is no photons there

$$E_n = \hbar\omega(n+\tfrac{1}{2}) \qquad E_0 = \tfrac{1}{2}\hbar\omega$$

Thus total zero-point energy $E = \sum_{mode} \frac{1}{2}\hbar\omega \to \infty$

Simple solution → renormalization, we are going to count energy from this zero point

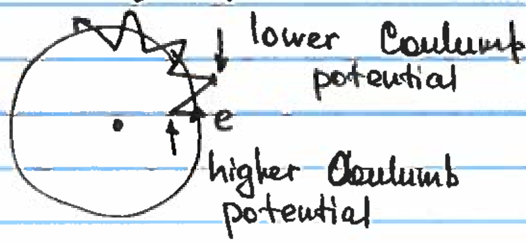
However, we still have non-vanishing <u>fluctuations</u> in e-m field amplitude → their effect is directly observable

- Spontaneous emission: an electron in excited states interacts with fluctuating vacuum fields that cause it to change its state, emitting thermal radiation

- Since there is always an uncertainty in measurable electric field ($\hat{a}$ and $\hat{a}^+$ do not commute), all optical measurements are fundamentally limited ~~by~~ in precision.

Lamb shift

Experimentally $2S_{1/2}$ and $2P_{1/2}$ states in Hydrogen atom are split by $\sim 1\,GHz$.
No semiclassical theory can explain this shift. It originates from the interaction of an electron with vacuum electric field



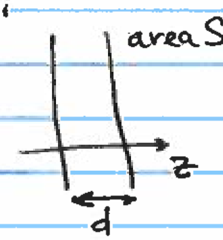
$$\Delta E_S = \frac{1}{6}\langle \delta r^2 \rangle \cdot 4\pi\varepsilon_0 \, |\psi(r=0)|^2$$

and zero for all $\ell \neq 0$ states

Casimir force

Two parallel conducting plates attract when close ($\sim 100\,nm$) to each other

Zero point energy $E_{ZPE} = \sum_{modes} \frac{1}{2}\hbar\omega_i$



Free space – all ω_i are possible
Two parallell conducting plates
$k_z = \pi n/d \qquad \omega_n = c k_n = \sqrt{k_x^2 + k_y^2 + (\pi n/d)^2}$

Inside the "cavity"

$$E^{(in)}_{ZPE} = \int dk_x\, dk_y \sum_n \frac{1}{2}\hbar c \sqrt{k_x^2 + k_y^2 + (\pi n/d)^2}$$

Outside the "cavity"

$$E^{(out)}_{ZPE} = \int dk_x\, dk_y\, dk_z \cdot \frac{1}{2}\hbar c \sqrt{k_x^2 + k_y^2 + k_z^2}$$

"Extra" energy inside the cavity

$$U = E^{(in)}_{ZPE} - E^{(out)}_{ZPE} = \left\{ \begin{array}{c} \text{long} \\ \text{calculations} \end{array} \right\} = \frac{\pi^2 \hbar c}{720 d^3} \cdot S$$

Casimir force per unit area

$$F = -\frac{1}{S}\frac{dU}{dd} = -\frac{\pi^2 \hbar c}{240\, d^4}$$ attractive force