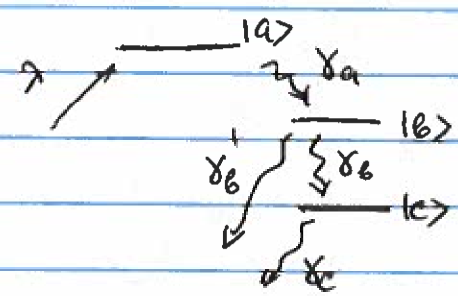
Midterm 1 solutions

1.



$$\dot\rho_{aa} = \lambda - \rho_{aa}\gamma_a$$
$$\dot\rho_{bb} = \rho_{aa}\gamma_a - \rho_{bb}(\gamma_b + \gamma_b')$$
$$\dot\rho_{cc} = \rho_{bb}\gamma_b - \rho_{cc}\gamma_c$$

a) Steady state case $\dot\rho_{ii} = 0$

$$\gamma_a \rho_{aa} = \lambda$$
$$\rho_{bb} = \frac{\lambda}{\gamma_b + \gamma_b'}$$
$$\rho_{cc} = \frac{\gamma_c}{\gamma_b}\rho_{bb} = \frac{\gamma_c}{\gamma_b}\frac{\lambda}{\gamma_b + \gamma_b'}$$

b) For a two-level system

$$\dot\rho_{bc} = -(\gamma_{bc} - i\Delta)\rho_{bc} + i\frac{\wp_{bc}\mathcal{E}}{2\hbar}(\rho_{bb} - \rho_{cc})$$
$$\gamma_{bc} = \frac{(\gamma_b + \gamma_b') + \gamma_c}{2} \qquad \rho_{bb} - \rho_{cc} \to \cancel{\rho_{bb}}\frac{\lambda}{\gamma_b + \gamma_b'}\left(1 - \frac{\gamma_c}{\gamma_b}\right)$$

$\dot\rho_{bc} = 0$
$$\rho_{bc} = i\frac{\wp_{bc}\mathcal{E}}{2\hbar}\frac{\rho_{bb} - \rho_{cc}}{\gamma_{bc} - i\Delta} = i\Omega\frac{\rho_{bb} - \rho_{cc}}{\gamma_{bc} - i\Delta}$$

$$\chi(\Delta) = \frac{\wp_{bc}\rho_{bc}}{\varepsilon_0 \mathcal{E}/2} = i\frac{\wp_{bc}^2}{\hbar\varepsilon_0}\frac{\rho_{bb} - \rho_{cc}}{\gamma_{bc} - i\Delta}$$

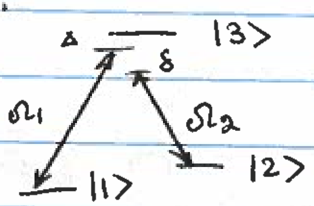
c) Absorption coefficient $\alpha(\Delta) = \frac{k}{2}\,\mathrm{Im}\,\chi(\Delta)$

$$\alpha(\Delta) = \frac{\wp_{bc}^2}{\hbar\varepsilon_0}(\rho_{bb} - \rho_{cc})\frac{\gamma_{bc}}{\gamma_{bc}^2 + \Delta^2}$$

Amplification: $\alpha(\Delta) < 0 \implies \rho_{bb} - \rho_{cc} < 0 \implies \left(1 - \frac{\gamma_c}{\gamma_b}\right) < 0$

$$\boxed{\gamma_b < \gamma_c}$$

2.



a) We expect the dark state to exist only for zero detunings $\Delta = 0 \quad \delta = 0$

$$\hat{H} = \hbar\Omega_1 |3\rangle\langle 1| + \hbar\Omega_2 |3\rangle\langle 2| + \text{c.c}$$

$$\hat{H}|D\rangle = 0$$

$$|D\rangle = \frac{1}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}} \left(\Omega_1|2\rangle - \Omega_2|1\rangle\right)$$

If $\Omega_1 = \Omega_2$ $\quad |D\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |1\rangle)$

If $\Omega_1 = |\Omega_2| e^{i\theta}$ $\quad |D\rangle = \frac{1}{\sqrt{2}}(|2\rangle - e^{i\theta}|1\rangle)$

b) Maxwell-Bloch equations (from the class notes)

$$\begin{cases} \dot{\rho}_{11} = \Gamma\rho_{33} - i\Omega_1\rho_{13} + i\Omega_1^*\rho_{31} \\ \dot{\rho}_{22} = \Gamma\rho_{33} - i\Omega_2\rho_{23} + i\Omega_2^*\rho_{32} \\ \dot{\rho}_{33} = -\dot{\rho}_{11} - \dot{\rho}_{22} \qquad \rho_{11} + \rho_{22} + \rho_{33} = 1 \\ \\ \dot{\rho}_{21} = -(\gamma - i\delta)\rho_{21} + i\Omega_2^*\rho_{31} - i\Omega_1\rho_{23} \\ \dot{\rho}_{31} = -(\Gamma - i\Delta)\rho_{31} + i\Omega_2\rho_{21} + i\Omega_1(\rho_{11} - \rho_{33}) \\ \dot{\rho}_{32} = -(\Gamma - i(\Delta+\delta))\rho_{32} + i\Omega_2(\rho_{22} - \rho_{33}) + i\Omega_1\rho_{12} \end{cases}$$

c) According to the dark state

$\rho_{11} = \rho_{22} = 1/2 \qquad \rho_{33} = 0 \quad , \quad \Delta = 0$

Since the populations are given, we only need to find the coherences

$$\dot{\rho}_{21} = 0 = -(\gamma - i\delta)\rho_{21} + i\Omega_2^*\rho_{31} - i\Omega_1\rho_{23}$$

$$\dot{\rho}_{31} = 0 = -\Gamma\rho_{31} + i\Omega_2\rho_{21} + \tfrac{1}{2} i\Omega_1$$

$$\dot{\rho}_{32} = 0 = -(\Gamma - \cancel{i\delta})\rho_{32} + i\tfrac{1}{2}\Omega_2 + i\Omega_1\rho_{12}$$

neglect for small δ

$$\rho_{31} = \frac{1}{\Gamma}\left(i\Omega_2\rho_{21} + \frac{1}{2} i\Omega_1\right)$$

$$\rho_{32} = \frac{1}{\Gamma}\left(i\Omega_1\rho_{12} + \frac{1}{2} i\Omega_2\right); \quad \rho_{23} = \frac{1}{\Gamma}\left(-i\Omega_1^*\rho_{21} - i\frac{1}{2}\Omega_2^*\right)$$

$$-(\gamma + i\delta)\rho_{21} + \frac{i\Omega_2^*}{\Gamma}\left(i\Omega_2\rho_{21} + \frac{1}{2} i\Omega_1\right) - \frac{i\Omega_1}{\Gamma}\left(-i\Omega_1^*\rho_{21} - i\frac{1}{2}\Omega_2^*\right)$$

$$-\left[\gamma - i\delta + \frac{|\Omega_2|^2 + |\Omega_1|^2}{\Gamma}\right]\rho_{21} - \frac{i\Omega_1\Omega_2^*}{\Gamma} = 0$$

$$\rho_{21} = -\frac{\Omega_1\Omega_2^*}{\Gamma(\gamma - i\delta) + 2|\Omega|^2} \qquad |\Omega| = |\Omega_1| = |\Omega_2|$$

$$\rho_{31} = \frac{i\Omega_1}{\Gamma}\left[\frac{1}{2} - \frac{|\Omega|^2}{\Gamma(\gamma - i\delta) + 2|\Omega|^2}\right]$$

$$\rho_{32} = \frac{i\Omega_2}{\Gamma}\left[\frac{1}{2} - \frac{|\Omega|^2}{\Gamma(\gamma + i\delta) + 2|\Omega|^2}\right]$$

~~Since Ω_1 and Ω_2 represent the same field, ...~~

$$\chi_1(\Delta=0) = i\frac{\wp_{13}^2}{\varepsilon_0\hbar\Gamma}\left[\frac{1}{2} - \frac{|\Omega|^2}{\Gamma(\gamma-i\delta)+2|\Omega|^2}\right] = i\frac{\wp_{13}^2}{2\varepsilon_0\hbar\cancel{\Gamma}}\frac{\cancel{\Gamma}(\gamma-i\delta)}{\Gamma(\gamma-i\delta)+2|\Omega|^2}$$

$$\alpha_1(\Delta=0) = \frac{k}{2}\,\mathrm{Im}\,\chi_1 = \frac{k\wp_{13}^2}{4\varepsilon_0\hbar\Gamma}\;\frac{-\delta^2+\gamma(\gamma+2|\Omega|^2/\Gamma)}{(\gamma+\frac{2|\Omega|^2}{\Gamma})^2+\delta^2}$$

similarly

$$\chi_2(\Delta=0) = \frac{i\wp_{23}^2}{\varepsilon_0\hbar\Gamma}\left[\frac{1}{2} - \frac{|\Omega|^2}{\Gamma(\gamma+i\delta)+2|\Omega|^2}\right] = i\frac{\wp_{23}^2}{2\varepsilon_0\hbar}\frac{\gamma+i\delta}{\Gamma(\gamma+i\delta)+2|\Omega|^2}$$

$$\alpha_2(\Delta=0) = \frac{k}{2}\,\mathrm{Im}\,\chi_2 = \frac{k\wp_{23}^2}{4\varepsilon_0\hbar\Gamma}\;\frac{-\delta^2+\gamma(\gamma+2|\Omega|^2/\Gamma)}{(\gamma+\frac{2|\Omega|^2}{\Gamma})^2+\delta^2}$$

Two fields are absorbed by the same amount

d) The refractive indices $n = 1 + \frac{\mathrm{Re}(\chi)}{2}$

$$n_1 = 1 + \frac{\wp_{13}^2}{4\varepsilon_0\hbar\Gamma}\;\frac{\delta\;2|\Omega|^2/\Gamma}{(\gamma+\frac{2|\Omega|^2}{\Gamma})^2+\delta^2}$$

$$n_2 = 1 - \frac{\wp_{23}^2}{4\varepsilon_0\hbar\Gamma}\;\frac{\delta\;2|\Omega|^2/\Gamma}{(\gamma+\frac{2|\Omega|^2}{\Gamma})^2+\delta^2}$$

The two field have the opposite change in the refractive index since $\delta = \omega_1 - \omega_2$

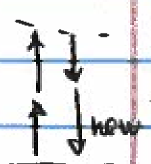
3. $\lambda_1 = 632.8\,nm$ $\qquad \omega_{1,2} = \dfrac{2\pi c}{\lambda_{1,2}}$

$\lambda_2 = 388\,nm$

Four-wave mixing $\quad E_{new} \propto \chi^{(3)} E^3$

all possible combinations of frequencies

ω_{new}	λ_{new}
$3\omega_1$	$\lambda_1/3 = 211\,nm$
$2\omega_1 + \omega_2$	$\left(\frac{2}{\lambda_1} + \frac{1}{\lambda_2}\right)^{-1} = \dfrac{\lambda_1\lambda_2}{2\lambda_2 + \lambda_1} = 174\,nm$
$2\omega_2 + \omega_1$	$\left(\frac{1}{\lambda_1} + \frac{2}{\lambda_2}\right)^{-1} = \dfrac{\lambda_1\lambda_2}{2\lambda_1 + \lambda_2} = 148\,nm$
$3\omega_2$	$\lambda_2/3 = 129\,nm$



$2\omega_1 - \omega_2$	$\left(\dfrac{1}{2/\lambda_1 - 1/\lambda_2}\right) = \dfrac{\lambda_1\lambda_2}{2\lambda_2 - \lambda_1} = 1714.5\,nm$
$2\omega_2 - \omega_1$	$\dfrac{1}{2/\lambda_2 - 1/\lambda_1} = \dfrac{\lambda_1\lambda_2}{2\lambda_1 - \lambda_2} = 279.8\,nm$

+ orignal λ_1, λ_2