Homework 3 solutions

1. Let's calculate the energy inside a cavity of volume V, containing one photon

$$\langle \text{energy} \rangle = \langle 1 | \int_V \varepsilon_0 \hat{E}^2 dV | 1 \rangle =$$

$$= \frac{\hbar\omega}{V} \int_V \sin^2 kz \, dV \left(\langle 1 | \hat{a}\hat{a}^+ | 1 \rangle + \langle 1 | \hat{a}^+ \hat{a} | 1 \rangle \right) =$$

$$= \underbrace{\frac{1}{2}\hbar\omega}_{\text{zero-point energy}} + \overbrace{\hbar\omega}^{\text{energy of a photon}}$$

What should be the classical amplitude of a EM field to have the same energy?

$$u_{EM} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0} B^2 = \varepsilon_0 E^2$$

$$\text{energy} = \int_V \overline{u_{em}} \, dV = \varepsilon_0 E^2 V = \hbar\omega$$

$$E = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}$$

2. $$\langle \alpha | \beta \rangle = e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{n,n'=0}^{\infty} \frac{(\alpha^*)^n \beta^{n'}}{\sqrt{n! \, n'!}} \overbrace{\langle n | n' \rangle}^{=\delta_{nn'}} =$$

$$= e^{-\frac{|\alpha|^2+|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha^*\beta)^n}{n!} = e^{-\frac{|\alpha|^2+|\beta|^2}{2}} e^{\alpha^*\beta}$$

3. $$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Probability to measure one photon $P_1 = e^{-|\alpha|^2} |\alpha|^2$

—— " —— two photons $P_2 = e^{-|\alpha|^2} \frac{|\alpha|^4}{2}$

$$p \leq \frac{P_2}{P_1} = \frac{|\alpha|^2}{2} \Rightarrow |\alpha|^2 \geq 2p$$

$|\alpha|^2$ is the average photon number in a coherent state

4. $|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\bar{\alpha}\rangle)$ — Schrodinger cat

$$\langle\psi|\hat{n}|\psi\rangle = \frac{1}{2}\left[\langle\alpha|\hat{n}|\alpha\rangle + \langle-\alpha|\hat{n}|-\alpha\rangle + \langle\alpha|\hat{n}|-\alpha\rangle + \langle-\alpha|\hat{n}|\alpha\rangle\right] =$$

$$= \frac{1}{2}\left[2|\alpha|^2 + |\alpha|^2(\langle-\alpha|\alpha\rangle + \langle\alpha|-\alpha\rangle)\right] \approx |\alpha|^2$$

From the problem 2: $\langle-\alpha|\alpha\rangle = e^{-2|\alpha|^2} \ll 1$ for $|\alpha|\geq 1$

$$\langle\psi|\hat{n}|\psi\rangle = \left[|\alpha|^2 + |\alpha|^2 e^{-2|\alpha|^2}\right] \approx |\alpha|^2$$

$$\langle\psi|\hat{E}_x|\psi\rangle = \langle\psi|\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}(\hat{a}e^{-i\omega t} + \hat{a}^+ e^{i\omega t})|\psi\rangle$$

$$\langle\psi|\hat{a}|\psi\rangle = \langle\alpha|\hat{a}|\alpha\rangle + \langle-\alpha|\hat{a}|-\alpha\rangle + \langle\alpha|\hat{a}|-\alpha\rangle + \langle-\alpha|\hat{a}|\alpha\rangle =$$

$$= \alpha + (-\alpha) - \alpha\langle\alpha|\alpha\rangle + \alpha\langle-\alpha|\alpha\rangle = 0$$

$$\langle E\rangle = 0$$

Two components of the schrodinger's cat state have opposite phases, and thus iterfer.

5) $|\alpha,1\rangle = \mathcal{N}\hat{a}^+|\alpha\rangle$

Normalization $\langle\alpha,1|\alpha,1\rangle = \mathcal{N}^2\langle\alpha|\hat{a}\hat{a}^+|\alpha\rangle =$

$$= \mathcal{N}^2\langle\alpha|(\hat{a}^+\hat{a}+1)|\alpha\rangle = \mathcal{N}\langle\alpha|\hat{n}+1|\alpha\rangle = \mathcal{N}(|\alpha|^2+1) = 1$$

$$\mathcal{N} = 1/\sqrt{|\alpha|^2+1}$$

$$|\alpha,1\rangle = \mathcal{N}\hat{a}^+|\alpha\rangle = \mathcal{N}e^{-|\alpha|^2/2}\hat{a}^+\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle =$$

$$= \mathcal{N}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n+1}\,|n+1\rangle =$$

$$= \mathcal{N}\sum_{n=1}^{\infty} n\frac{\alpha^{n-1}}{\sqrt{n!}}|n\rangle \qquad P_n = n^2\frac{|\alpha|^{2n-2}}{|\alpha|^2+1}\frac{1}{n!} =$$

$$P_n = \frac{n}{(n-1)!}\frac{|\alpha|^{2n-2}}{|\alpha|^2+1} \qquad n=1,2,\ldots \quad \text{no } n=0 \text{ component}$$

6. $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |10\rangle)$ $\quad$ $\hat{a}|\psi\rangle = |\psi'\rangle = |9\rangle$

(after renormalization)

$\bar{n} = \langle\psi|\hat{n}|\psi\rangle = 5$ $\quad$ $\bar{n}' = \langle\psi'|\hat{n}\psi'\rangle = 9$

May look surprizing, but if we know that exacty one photon was eliminated, we know that the state had 10 photons, not zero. Thus, by permitting such measurement we collapse the superposition!

7. $\hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})}$ $\quad$ $\xi = re^{i\theta}$

need to find r and θ

Amplitude squeezing $\to \Delta\hat{X}_1$ is ~~[illegible]~~ squeezed

$\theta = 0$

Measured squeezing in dB $= -10\log_{10}\frac{\langle\Delta X_1\rangle_{sqz}}{\langle\Delta X_1\rangle_{vac}}$

$5\,dB = -10\log_{10} e^{-2r} = -10\log_{10} e^{-2r}$

$e^{-2r} = 0.316$ $\quad$ $r = 0.58$

$\hat{S} = e^{-0.29(\hat{a}^2 - (\hat{a}^+)^2)}$