Spontaneous emission

Caused by interaction of an atom in an excited state with e-m <u>vacuum</u> field

$$\hat V = -e\vec r\cdot\vec E = -ez\mathcal{E}e^{i\omega t} + c.c.$$

———— $|2\rangle$

Interaction of a two-level atom with e-m field

———— $|1\rangle$

$$\hat V = \hbar\begin{pmatrix}0 & \Omega e^{i\omega t}\\ \Omega^* e^{-i\omega t} & 0\end{pmatrix}$$

Can re-write $\hat V = \hbar\Omega e^{i\omega t}|2\rangle\langle 1| + \hbar\Omega^* e^{-i\omega t}|1\rangle\langle 2|$

$\Omega = \dfrac{\wp_{12}\mathcal{E}}{\hbar}$ - Rabi frequency

For a single-mode quantum e-m field

$$\mathcal{E}\to\sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}\,\hat a$$

$$\Omega \to \underbrace{\left[\frac{\wp_{12}}{\hbar}\sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}\right]}_{g_{12}}\hat a = g_{12}\,\hat a$$

g_{12} - a Rabi frequency of a single photon

$$\hat V = \underbrace{\hbar g\hat a\,|2\rangle\langle 1|}_{\text{a photon absorbed, atom exited } 1\to 2} + \underbrace{\hbar g^*\hat a^+|1\rangle\langle 2|}_{\text{a photon emitted, atom is de-excited}}$$

Various vacuum modes: all possible $\{\vec{k}, \omega\}$

$$\hat{V} = \sum_k \left(\hbar g_k \hat{a}_k |2\rangle\langle 1| e^{i(\omega_{12}-\omega_k)t} + \hbar g_k^* \hat{a}_k^+ |1\rangle\langle 2| e^{-i(\omega_{12}-\omega_k)t} \right)$$

State of the system: $|\text{atom, all photon modes}\rangle$

Initial state $|2, \emptyset\rangle$

Intermediate state: $c_2(t)|2, \emptyset\rangle + \sum_k c_{1k}(t)|1, 1_k\rangle$

one photon in one of the modes

$$|\dot{\psi}(t)\rangle = -\frac{i}{\hbar} \hat{V} |\psi(t)\rangle$$

$$\dot{c}_2 = -i \sum_k g_{\vec{k}} e^{i(\omega_{12}-\omega_k)t} c_{1k}(t)$$

$$\dot{c}_{1k} = -i g_{\vec{k}}^* e^{-i(\omega_{12}-\omega_k)t} c_2(t) \Rightarrow c_{1\vec{k}} = -i g_{\vec{k}}^* \int_0^t e^{-i(\omega_{12}-\omega_k)t'} \cdot c_2(t')dt'$$

$$\dot{c}_2(t) = -\sum_{\vec{k}} |g_{\vec{k}}|^2 \int_0^t dt' e^{i(\omega_{12}-\omega_k)(t-t')} c_2(t')$$

$$|g_{\vec{k}}|^2 = \frac{p_{12}^2 \omega_k}{2\hbar\varepsilon_0 V} \cos^2\theta \qquad |\vec{k}| = \frac{\omega_k}{c}$$

$$\sum_{\vec{k}} \longrightarrow \frac{2V}{(2\pi)^3} \int d^3\vec{k} = \frac{2V}{(2\pi)^3 c^3} \int_0^{2\pi} d\varphi \int_0^{\pi} d(\cos\theta) \int_0^{\infty} \omega_k^2 d\omega_k$$

$$\dot{c}_2(t) = -\frac{4p_{12}^2}{(2\pi)^2 6\hbar\varepsilon_0 c^3} \int_0^{\infty} \omega_k^3 d\omega_k \int_0^t dt' e^{i(\omega_{12}-\omega_k)(t-t')} c_2(t')$$

$$\approx -\frac{4p_{12}^2\omega_{12}^2}{(2\pi)^2 6\hbar\varepsilon_0 c^3} \int_0^t c_2(t')dt' \cdot \underbrace{\int_0^{\infty} e^{i(\omega_{12}-\omega)(t-t')} d\omega'}_{=\pi\delta(t-t')}$$

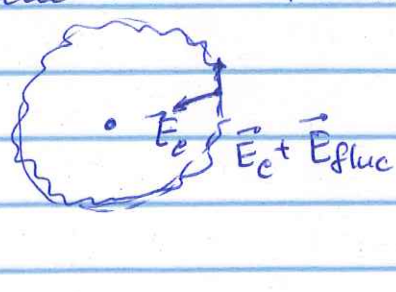
$$\dot{c}_2(t) = -\frac{\Gamma}{2} c_2(t) \qquad \boxed{\Gamma = \frac{1}{4\pi\varepsilon_0} \frac{4\omega^3 p_{12}^2}{3\hbar c^3}} \text{ spontaneous decay rate}$$

$$|c_2(t)|^2 = e^{-\Gamma t} \quad \text{spontaneous decay}$$

Lamb shift

We have seen before that for any semi-classical treatment (i.e. quantized atom + classical E-M field) there is a degeneracy b/w $2S_{1/2}$ and $2P_{1/2}$ states in a hydrogen atom.
However, these two levels are nondegenerate in reality, with the frequency splitting of 1057 MHz — Lamb shift
First measured experimentally, it can only be explained by QED and vacuum fluctuations
Full relativistic calculations are in excellent agreement with the experiment; however, here I give a heuristic derivation, that captures the essence of the effect.

Vacuum fluctuations will perturb the orbit of an electron from its standard orbit due to the Coulumb potential $-e^2/r$ due to the proton, $r \to r + \delta r$, where δr is the fluctuation in the position of the electron due to fluctuating field



Energy shift

$$\delta V = V(\vec{r} + \delta\vec{r}) - V(\vec{r}) =$$
$$= \delta\vec{r}\,\nabla V + \frac{1}{2}(\delta\vec{r}\cdot\nabla)^2 V(\vec{r}) + \dots$$

For fluctuations $\langle \delta\vec{r} \rangle = 0$ so

$$\langle \Delta V \rangle = \frac{1}{2}\langle (\delta\vec{r}\,\nabla)^2 V(\vec{r}) \rangle = \frac{1}{6}\langle (\delta\vec{r})^2 \rangle_{vac} \langle \nabla^2 V(r) \rangle$$

since fluctuations are ~~anis~~ isotropic

< For these calculations I will be using SI system, following my source (Scully & Zubairy, "Quantum optics")

$$V = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$\langle \nabla^2 V(\vec{r}) \rangle = \langle \nabla^2 \left(-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \right) \rangle$$

For 2s state

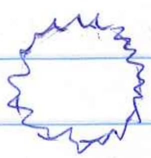
$$\langle \nabla^2 V(r) \rangle_{2s} = -\frac{e^2}{4\pi\varepsilon_0} \oint \langle 2s | \nabla^2 \frac{1}{r} | 2s \rangle = \qquad \nabla^2 \frac{1}{r} = -4\pi\,\delta(\vec{r})$$

$$= \frac{e^2}{\varepsilon_0} |\psi_{2s}(0)|^2 = \frac{e^2}{8\pi\varepsilon_0 a_0^3} > 0$$ 2s state is shifted up

For 2p state $\psi_{2p}(0) = 0$ (since $\psi_{2p}(r) \sim r$), so in non-relativistic approximation there is no shift for 2p state

Next, we need to estimate the variance of the electron orbit fluctuation $\langle (\delta r)^2 \rangle$

Classical eqn of motion



$$m \frac{d^2}{dt^2}(\delta r)_{\vec{k}} = -e E_{\vec{k}}$$ valid for high frequencies $\omega \gtrsim \frac{\pi c}{a_0}$

where now $\vec{E}_k$ is a single mode of a vacuum field of wavevector $\vec{k}$

$$E_k = \sqrt{\frac{\hbar c k}{2\varepsilon_0 V}} \left(\hat{a}_k e^{i\vec{k}\vec{r} - i\omega t} + \hat{a}_k^{+} e^{-i\vec{k}\vec{r} + i\omega t} \right)$$

$$m \frac{d^2}{dt^2} (\delta \hat{r})_k = -e \, E_k \quad (\dots\, e^{-i\omega_k t} + \dots\, e^{i\omega_k t})$$

$$\delta \hat{r}(t)|_k = \delta \hat{r}(0)|_k e^{-i\omega_k t} + c.c.$$

$$m(-i\omega)^2 (\delta \hat{r})_k = -e E_k$$

$$\boxed{k = \frac{\omega_k}{c}}$$

$$(\delta \hat{r})_k = + \frac{e \hat{E}_k}{m\omega^2} = \frac{e \hat{E}_k}{m c^2 k^2}$$

Assuming all modes are independent $\langle (\delta r_k)(\delta r_{k'}) \rangle = 0$, we can write

$$\langle (\delta r)^2 \rangle = \sum_k \frac{e^2}{m^2 c^4 k^4} \langle E_k^2 \rangle$$

where $\langle E_k^2 \rangle = \langle 0 | E_k^2 | 0 \rangle = \frac{\hbar c k}{2\varepsilon_0 V} \langle 0 | \hat{a}_k^2 e^{-2i\omega t + 2i\vec{k}\vec{r}} +$

$$+ \hat{a}_k^{\dagger 2} e^{+2i\omega t - 2i\vec{k}\vec{r}} + \underbrace{\hat{a}_k \hat{a}_k^\dagger}_{\text{only non-zero term,}} + \hat{a}_k^\dagger \hat{a}_k |0\rangle$$

$$a_k |0\rangle = 0, \quad \hat{a}_k \hat{a}_k^\dagger |0\rangle = 1 \cdot |0\rangle$$

$$\langle E_k^2 \rangle = \frac{\hbar c k}{2\varepsilon_0 V}$$

$$\langle (\delta r)^2 \rangle = \sum_k \frac{e^2}{m^2 c^4 k^4} \frac{\hbar c k}{2\varepsilon_0 V} = \sum_k \frac{\hbar e^2}{2\varepsilon_0 V} \frac{1}{m^2 c^3 k^3}$$

For large volume we can replace $\sum_k \to \int d^3\vec{k}$ as we discuss, $\delta \vec{r}$ is ~~ani~~ isotropic, so no angular dependence. Thus

$$\langle (\delta r)^2 \rangle = \frac{V}{(2\pi)^3} \int 4\pi k^2 dk \cdot \frac{\hbar e^2}{2\varepsilon_0 V m^2 c^3} \frac{1}{k^3} = \frac{1}{2\varepsilon_0 \pi^2} \left(\frac{e^2}{\hbar c}\right) \left(\frac{\hbar}{mc}\right)^2 \times$$

$$\times \int \frac{dk}{k} \quad \leftarrow \text{diverges if integrated } \int_0^\infty$$

$k_{min} = \frac{\pi}{a_0} = \frac{me^2}{4\varepsilon_0 \hbar^2}$

$\frac{k_{max}}{k_{min}} = \frac{mc}{\hbar} \cdot \frac{4\varepsilon_0 \hbar^2}{me^2}$

However, we started with assuming only high frequency fluctuations $\omega_{min} = \frac{\pi c}{a_0}$, $k_{min} = \frac{\pi}{a_0}$. We also have to stay within non-relativistic approximation $\lambda > \lambda_{compt}$, $k < mc/\hbar$ to neglect magnetic effects ($v/c = p/mc = \frac{\hbar k}{mc} \leq 1$)

$$k_{max} = mc/\hbar$$

$$\int_{k_{min}}^{k_{min}} \frac{dk}{k} = \ln k \Big|_{k_{min}}^{k_{max}} = \ln \left| \frac{k_{max}}{k_{min}} \right|$$

$$\langle (\delta r)^2 \rangle \approx \frac{1}{2\varepsilon_0 \pi^2} \left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{mc}\right)^2 \ln\left(\frac{4\varepsilon_0 \hbar c}{e^2}\right)$$

Total shift of 2S state

$$\langle \Delta V \rangle = \frac{4}{3} \frac{e^2}{4\pi\varepsilon_0} \frac{e^2}{4\pi\varepsilon_0 \hbar c} \left(\frac{\hbar}{mc}\right)^2 \frac{1}{8\pi a_0^3} \ln\left(\frac{4\varepsilon_0 \hbar c}{e^2}\right)$$

$\Delta V \simeq 1\,GHz$, rather close to accurate 1057 MHz.

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Casimir effect

E-M field Hamiltonian

$$\hat{H}_{EM} = \sum_{\vec{k},\lambda} \frac{1}{2}\hbar\omega_k \left[\hat{a}_k \hat{a}^+ + \hat{a}_k^+ \hat{a}\right]$$

The state with zero photons $|0\rangle = \prod_k |0\rangle_k$ for all $\vec{k}$

$$H|0\rangle = \sum_{\vec{k},\lambda} \frac{1}{2}\hbar\omega_k \left[\hat{a}_k^+ \hat{a} + \underline{\underline{\hat{a}\hat{a}^+}}\right] |0_k\rangle_k \cdot \prod_{k' \neq k} |0_k\rangle_{k'} =$$

$$= \frac{1}{2} \sum_{\vec{k},\lambda} \hbar\omega_k \underbrace{\hat{a}\hat{a}^+ |0_k\rangle_k}_{=1\cdot|0\rangle_k} \cdot \prod_{\vec{k}' \neq k} |0_k\rangle_{k'} = \frac{1}{2}\sum_{\vec{k},\lambda} \hbar\omega_k |0\rangle$$

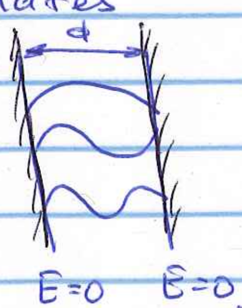
$$= E_0 |0\rangle$$

E_0 - zero-point energy

$$E_0 = \frac{1}{2}\sum_{\vec{k},\lambda} \hbar\omega_k = \sum_{\vec{k}} \hbar\omega_k$$

~~vacuum~~ The energy of vacuum, which is technically infinite $\rightarrow$ need to renormalize our energy "zero"

However, what if this energy is modified? Space b/w two infinite ~~pla~~ conducting plates



E=0 E=0

on the boundary

only ~~discrete~~ some modes can exist inside this space

$$n\lambda/2 = d = n\frac{\pi}{k_n} = d$$

$$k_n = \frac{n\pi}{d} \qquad n = 1, 2 \ldots$$

The vacuum energy is the sum of all modes, thus it will be modified if less modes exist b/w the plate

$$E_0\,(\text{free space}) = E_0(d=\infty) = \sum_{\vec{k}} \hbar\omega_k =$$

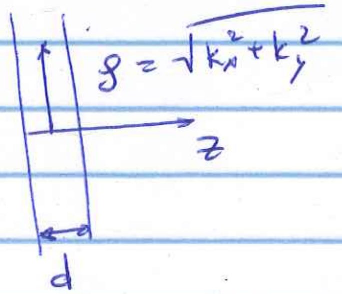
$$= \left(\frac{L}{\pi}\right)^2 \left(\frac{d}{\pi}\right) \int_0^\infty dk_x\, dk_y\, dk_z \; \hbar c \sqrt{k_x^2+k_y^2+k_z^2}$$

$$E_0(d) = \left(\frac{L}{\pi}\right)^2 \int_0^\infty dk_x\, dk_y \; \hbar c \sum_{n=1}^{\infty} \sqrt{k_x^2+k_y^2+\left(\frac{\pi n}{d}\right)^2}$$

Difference in vacuum energy $\cancel{\Delta E}\; U(d) = E_0(d) - E_0(\infty)$ is the finite difference b/w two infinite numbers.

However, the integral diverges for large k_x, k_y, and for those $\sqrt{k_x^2+k_y^2+\left(\frac{\pi n}{d}\right)^2} \approx \sqrt{k_x^2+k_y^2}$ so $\cancel{E_0(d)\big|_k \approx E_0(\infty)\big|_k}$ the two expressions become equivalent

Let's rewrite $U(d)$ and $E_0(d)$ using "cylindrical" coordinates for $\vec{k}$



$dk_x\, dk_y = 2\pi \rho\, d\rho$ (isotropic situation)

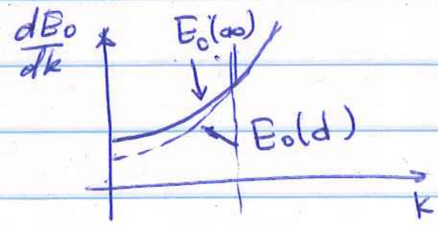
$$\sqrt{k_x^2+k_y^2+\left(\frac{\pi n}{d}\right)^2} = \sqrt{\rho^2+\left(\frac{\pi n}{d}\right)^2}$$

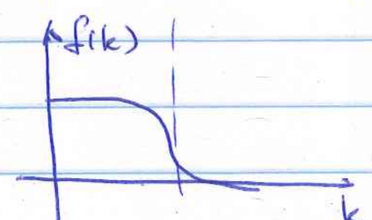
$(k_x, k_y \in [0,\infty])$

$$U(d) = 2\pi\hbar c \left(\frac{L}{\pi}\right)^2 \frac{1}{4}\int_0^\infty \rho\, d\rho \sum_n \sqrt{\rho^2 + \left(\frac{\pi n}{d}\right)^2} -$$

$$- 2\pi\hbar c \left(\frac{L}{\pi}\right)^2 \frac{d}{\pi}\cdot\frac{1}{4}\int_0^\infty \rho\, d\rho \int_0^\infty dk_z \sqrt{\rho^2 + k_z^2}$$

To avoid subtracting two infinite number, we can artificially introduce a cut-off function $f(k)$, such that $f(k) = 1$ for small k (that give non-zero contribution to $U(d)$) and $f(k) \to 0$ for large k (that cancel out at the end)





$$U(d) = \frac{1}{2}\pi\hbar c \left(\frac{L}{\pi}\right)^2 \int_0^\infty \rho\, d\rho \left\{ \sum_n f\left(\sqrt{\rho^2 + \left(\frac{\pi n}{d}\right)^2}\right) \sqrt{\rho^2 + \left(\frac{\pi n}{d}\right)^2} - \right.$$

$$\left. - \frac{d}{\pi}\int_0^\infty dk_z\, f\left(\sqrt{\rho^2 + k_z^2}\right)\sqrt{\rho^2 + k_z^2} \right.$$

~~Moving to dimensionless variables~~ $\cancel{x = \left(\frac{d}{\pi}\right)^2 \rho^2}$ ~~and~~ $\cancel{\varkappa = \left(\frac{d}{\pi}\right) k_z}$

$$\cancel{U(d) = \frac{1}{4}\pi\hbar c \left(\frac{L}{\pi}\right)^2 \left(\frac{\pi}{d}\right)^3 \int_0^\infty dx \left\{ \sum_n f\left(\sqrt{x + n^2}\,\frac{\pi}{d}\right)\sqrt{x + n^2} - \right.}$$

$$\cancel{\left. - \int_0^\infty d\varkappa\, f\left(\frac{\pi}{d}\sqrt{x + \varkappa^2}\right)\sqrt{x + \varkappa^2}\right.}$$

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Moving to a dimentionless variable for

$x = \frac{d}{\pi} k_z$

$$U(d) = \frac{1}{2} \pi \hbar c \left(\frac{L}{\pi}\right)^2 \int_0^\infty \rho d\rho \left\{ \sum_{n=0}^{\infty} f\left(\sqrt{\rho^2 + \left(\frac{\pi n}{d}\right)^2}\right) \sqrt{\rho^2 + \left(\frac{\pi n}{d}\right)^2} - \right.$$

$$\left. - \int_0^\infty dx\, f\left(\sqrt{\rho^2 + \left(\frac{\pi x}{d}\right)^2}\right) \sqrt{\rho^2 + \left(\frac{\pi x}{d}\right)^2} \right.$$

Substitution $\frac{\pi}{d} y = \sqrt{\rho^2 + \left(\frac{\pi x}{d}\right)^2} \Rightarrow y^2 = \frac{d^2 \rho^2}{\pi^2} + x^2$ (or n^2)

$$2y\,dy = \frac{d^2}{\pi^2} 2\rho\, d\rho \Rightarrow \rho\, d\rho = \frac{\pi^2}{d^2} y\, dy$$

$$U(d) = \frac{1}{2} \pi \hbar c \left(\frac{L}{\pi}\right)^2 \left(\frac{\pi}{d}\right)^3 \left\{ \sum_{n=0} \underbrace{\int_n^\infty y^2 dy\, f\left(\frac{\pi y}{d}\right)}_{F(n)} - \int_0^\infty dx \underbrace{\int_x^\infty y^2 dy\, f\left(\frac{\pi y}{d}\right)}_{F(x)} \right.$$

$\sum_n \rightarrow \frac{1}{2}$ for $n = 0$

$$F(x) = \int_x^\infty y^2 dy\, f\left(\frac{\pi y}{d}\right)$$

$$U(d) = \frac{\pi^2 \hbar c}{2 d^3} L^2 \left[\frac{1}{2} F(0) + \sum_{n=1}^\infty F(n) - \int_0^\infty F(x)\, dx \right]$$

$$\approx \frac{\pi^2 \hbar c}{2 d^3} L^2 \left[-\frac{1}{12} F'(0) + \frac{1}{720} F'''(0) + \ldots \right]$$

(Euler - Maclaurin summation)

$$F(x) = \int_x^\infty y^2 f\left(\frac{\pi y}{d}\right) dy \Rightarrow F'(x) = -x^2 f\left(\frac{\pi x}{d}\right) \Big|_{x=0} = 0$$

$x \to 0$

$$F''(x) = -2x f\left(\frac{\pi x}{d}\right) - \cancel{x^2 f'\left(\frac{\pi x}{d}\right)}$$

$$F''(x)\Big|_{x=0} = -2 f\left(\frac{\pi x}{d}\right) = -2$$

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$$U(d) = \frac{\pi^2 \hbar c}{2d^3} L^2 \left(\frac{-2}{720}\right) = -\frac{\pi^2 \hbar c}{720 d^3} L^2$$

Energy per unit area $\frac{U(d)}{L^2} = -\frac{\pi^2 \hbar c}{720 d^3}$

and force per unit area (Casimir force)

$$F_c = -\frac{1}{L^2}\frac{d\,U(d)}{d(d)} = -\frac{\pi^2 \hbar c}{240\, d^4}$$

Casimir force b/w a plane and a sphere

$$F_c^{(sphere)} = -\frac{\pi^3 R}{360}\left(\frac{\hbar c}{d^3}\right)$$