Time-dependent perturbation theory

Reminder: state time evolution in case of stationary hamiltonian

Time-independent Shrodinger eqn

$$\hat{H}\,|n(\vec{r})\rangle = E_n\,|n(\vec{r})\rangle$$

This is a simplified version of the general Shrodinger eqn

$$i\hbar\,\frac{\partial|n(\vec{r},t)\rangle}{\partial t} = \hat{H}\,|n(\vec{r},t)\rangle$$

where

$$|n(\vec{r},t)\rangle = \underbrace{e^{-i\frac{E_n t}{\hbar}}}_{\text{time dependence}}\,|n(\vec{r})\rangle$$

For any state $|\alpha\rangle \neq |n\rangle$

If $|\alpha\rangle = \sum c_n|n\rangle$ at $t=0$

then

$$|\alpha(t)\rangle = \sum \underbrace{c_n}_{\text{don't change in time}} e^{-iE_n t/\hbar}\,|n\rangle$$

If the hamiltonian describes a closed system (all interactions are contained within, no external influences), we can solve its time evolution using this approach. However, in practice its application is limited to very simple systems.

More realistically, we often want to describe open system, with a time-dependent interaction hamiltonian describing external environment

New hamiltonian

$$\hat{H} = \underset{\substack{\uparrow \\ \text{time-independent}}}{\hat{H}_0} + \hat{V}(t) \leftarrow \text{time dependent}$$

As before, the eigenfunctions and eigenvalues of H_0 are completely known.

Solving for $|\psi(t)\rangle \Rightarrow i\hbar \frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$

by assuming the solution in the form

$$|\psi(t)\rangle = \sum_n c_n(t)\,|n(t)\rangle = \sum_n c_n(t)\, e^{-\frac{i\cancel{t}E_n}{\hbar}}|n\rangle$$

similar to the previous, time-independent case, but now $c_n(t)$ are time-dependent because of $\hat{V}(t)$.

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = i\hbar\sum_n\left(\frac{\partial c_n}{\partial t}e^{-\frac{iE_nt}{\hbar}} \cancel{- \frac{iE_n}{\hbar}c_n e^{\frac{-iE_nt}{\hbar}}}\right)|n\rangle =$$

$$= \sum_n c_n e^{-iE_nt/\hbar}(\hat{H}_0 + \hat{V})|n\rangle =$$

$$= \cancel{\sum_n c_n e^{-iE_nt/\hbar} E_n|n\rangle} + \sum_{n'} c_{n'} e^{-iE_{n'}t/\hbar}\,\hat{V}|n'\rangle$$

$$i\hbar\frac{\partial c_n}{\partial t}e^{-iE_nt/\hbar} = \sum_{n'} c_{n'}\langle n|\hat{V}(t)|n'\rangle e^{-iE_{n'}t/\hbar}$$

$$i\hbar\frac{\partial c_n}{\partial t} = \sum_{n'} c_{n'} V_{nn'}\, e^{\frac{i(E_n - E_{n'})}{\hbar}t}$$

denoting $\omega_{nn'} = \frac{E_n - E_{n'}}{\hbar}$

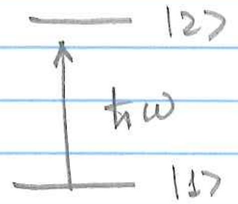
$$\boxed{i\hbar\frac{\partial c_n}{\partial t} = \sum_{n'} c_{n'} V_{nn'}(t)\, e^{i\omega_{nn'}t}}$$

exact solution

$\Downarrow$

complexity grows quickly with system size

Example: two-level system



Electromagnetic field

$$\vec{E}(t) = \vec{e}_z\, \mathcal{E} \cos\omega t =$$
$$= \frac{1}{2} \mathcal{E} \vec{e}_z \left(e^{i\omega t} + e^{-i\omega t}\right)$$

Recall from our treatment of
de-Stark effect

$$\hat{V} = -\frac{1}{2} e\mathcal{E} z \left(e^{i\omega t} + e^{-i\omega t}\right)$$

Assuming $|1\rangle$ and $|2\rangle$ are two states
with $\Delta l = 1$ and $\Delta m = 0$

$V_{11} = V_{22} = 0$ $\quad V_{12} = V_{21}^{*} = \langle 1|\hat{V}|2\rangle = \gamma\left(e^{i\omega t} + e^{-i\omega t}\right)$

if $\quad \gamma = \langle 1| -\frac{1}{2} e\mathcal{E} z |2\rangle$

Then we need to solve:

$\omega_{12} = \frac{E_1 - E_2}{\hbar} < 0$

$\omega_{21} = \frac{E_2 - E_1}{\hbar} > 0$

$$i\hbar \frac{\partial c_1}{\partial t} = c_2 \gamma \left(e^{i\omega t} + e^{-i\omega t}\right) e^{i\omega_{12} t}$$

$$i\hbar \frac{\partial c_1}{\partial t} = c_2 \gamma e^{i(\omega - \omega_{21})t} + c_2 \gamma e^{-i(\omega + \omega_{21})t}$$

For $\omega \approx \omega_{21}$ $\qquad e^{i(\omega - \omega_{21})t}$ – slowly varying term

$e^{i(\omega + \omega_{21})t} \sim e^{i2\omega_{21} t}$ – fast oscillations

For majority of observations we cannot capture such fast dynamics $\longrightarrow$ can neglect

This is equivalent to having

$$\hat{V} = \begin{pmatrix} 0 & \gamma e^{i\omega t} \\ \gamma e^{-i\omega t} & 0 \end{pmatrix}$$

$$i\hbar \dot{c}_1 = c_2 \gamma e^{i(\omega - \omega_{21})t}$$

$$i\hbar \dot{c}_2 = c_1 \gamma e^{-i(\omega - \omega_{21})t}$$

Can find exact solution (HW)
For now set $\omega = \omega_{21}$

$$i\hbar \dot{c}_1 = \gamma c_2 \quad \Rightarrow \quad \dot{c}_2 = \frac{i\hbar}{\gamma} \ddot{c}_1$$
$$i\hbar \dot{c}_2 = \gamma c_1$$

$$-\frac{\hbar^2}{\gamma} \ddot{c}_1 = \gamma c_1 \quad \Rightarrow \quad \ddot{c}_1 + \frac{\gamma^2}{\hbar^2} c_1 = 0$$

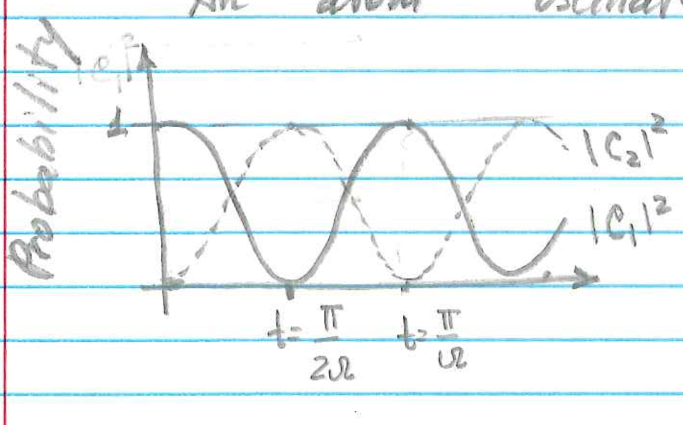
Now we need initial conditions
Let the system be at the state $|1\rangle$ at $t=0$

$$c_1(t=0) = 1, \qquad c_2(t=0) = 0$$

$$c_1(t) = \cos \frac{\gamma t}{\hbar} = \cos \Omega t$$

$$|c_2|^2 = 1 - |c_1|^2 \Rightarrow c_2 = \sin \Omega t$$

An atom oscillates b/w states 1 & 2



$$\Omega = \frac{\gamma}{\hbar} = \frac{\wp_{12} \mathcal{E}}{\hbar}$$

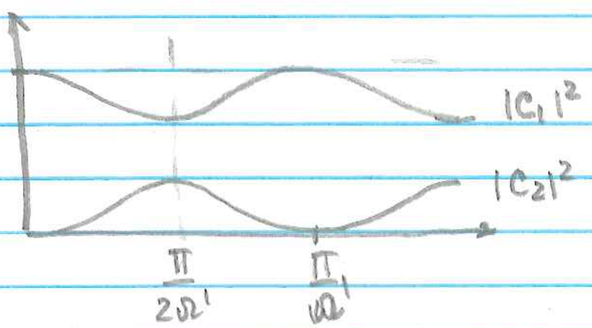
Rabi frequency

$\wp_{12}$ - dipole matrix element of a corresponding transition

For $\omega = \omega_{21} + \Delta$ $\qquad \Delta \ll \omega, \omega_{21}$

$$|C_2|^2 = \frac{\gamma^2}{\gamma^2 + \hbar^2\Delta^2/4} \sin^2 \Omega' t$$

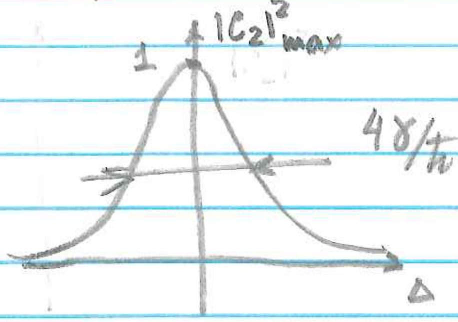
$$|C_1|^2 = 1 - |C_2|^2$$



$$\Omega' = \sqrt{\frac{\gamma^2}{\hbar^2} + \frac{\Delta^2}{4}}$$

generalized Rabi frequency

Amplitude as function of Δ



$$|C_2|^2_{max} = \frac{\gamma^2}{\gamma^2 + \hbar^2\Delta^2/4}$$

Amplitude peaks on resonance
Width proportional to the strength of perturbation