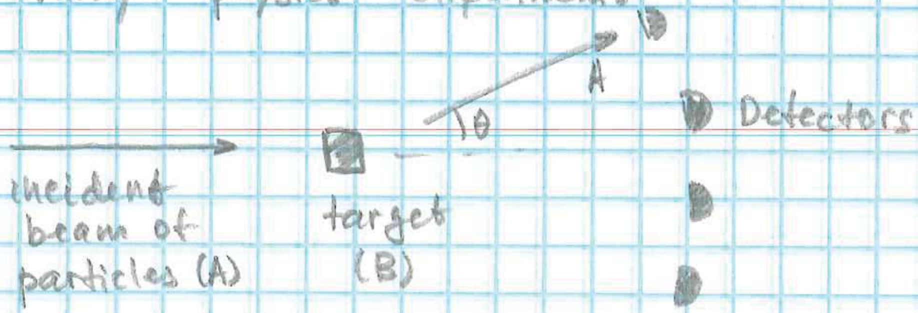
Scattering theory

Many physics experiments



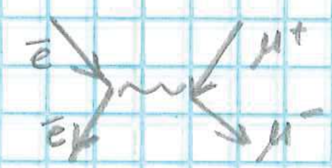
Possible outcomes

scattering

1. Elastic scattering:
   particle A bounces off target w/o changing internal structure of A or B
   (vision, $\bar{e}-\bar{e}$ scattering)

2. Inelastic scattering
   internal stucture of A or B changes;
   kinetic energy absorbed in collision
   (fluorescence, Raman scattering, $e^- + H \to e^- + H^*$)

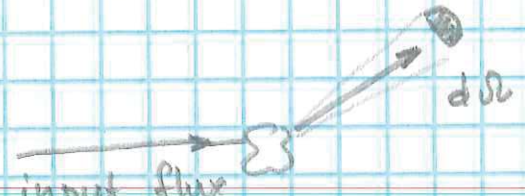
3. Rearrangement collisions
   outgoing particles are not A & B



We will focus on <u>elastic collisions</u>
Assumptions:

- particles A, B are spinless
- A, B have no internal structure (guarantee elactic collision)
- all interactions are approximated by a central potential $V(\vec{r})$
  (can alway be done by moving to CM reference frame)

General set-up



Number of particles hitting the detector

$$\frac{dn}{dt} = F \, \sigma(\theta,\varphi)\, d\Omega$$

$\uparrow$ differential scattering cross-section

It contains all the physics of the interaction

Total scattering cross-section

$$\sigma = \int \sigma(\theta,\varphi)\, d\Omega$$

Traditional units (in particle physics)
1 barn $= 10^{-24}\ cm^{-2}$

Assimptotic solution
Far enough from the scatterer $\rightarrow$ free particle

$$\frac{\hat{p}^2}{2m}\psi(\vec{r}) = E\,\psi(\vec{r})$$

We consider that all incoming particle have a given momentum $\vec{k}$ (choose $\vec{k} = (0,0,k)$)

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) = \frac{\hbar^2 k^2}{2m}\psi(\vec{r})$$

Assimptotic form of the wave function

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}}\left[e^{ikz} + \frac{f(\theta,\varphi)}{r}e^{ikr}\right]$$

From this $d\sigma = |f(\theta,\varphi)|^2 d\Omega$ (away from the region of the incoming beam)

Lippman - Schwinger equation

In elastic collisions the energy of the particle does not change

$E_{in} = E_{out}$

$\hat{V}=0 \qquad \hat{H}_0 |\psi_0\rangle = E|\psi_0\rangle$

Thus for $\hat{H} = \hat{H}_0 + \hat{V} \Rightarrow |\psi\rangle = |\psi_0\rangle + |\chi\rangle$

$$\hat{H}|\psi\rangle = \cancel{\hat{H}_0|\psi_0\rangle} + \hat{H}_0|\chi\rangle + \hat{V}|\psi\rangle = \cancel{E|\psi_0\rangle} + E|\chi\rangle$$

$$(E-\hat{H}_0)|\chi\rangle = \hat{V}|\psi\rangle$$

$$|\chi\rangle = \frac{1}{E-\hat{H}_0}\hat{V}|\psi\rangle$$

$$\boxed{|\psi\rangle = |\psi_0\rangle + \frac{1}{E-\hat{H}_0}\hat{V}|\psi\rangle}$$ Lippmann - Schwinger equation

Problem: a singularity!

Unlike in the time-independent perturbation theory we can't "project out" the states with momentum, equal to the initial momentum, because of the continuous spectrum.

We can deal with a singularity by "moving" it off-axis

$$\frac{1}{E_0-\hat{H}} \longrightarrow \frac{1}{E_0-\hat{H}\pm i\varepsilon}$$ [see Sakurai, Napolitano sec. 6.2]

Pertubative expansion

0: $|\psi\rangle = |\psi_0\rangle$

1: $|\psi\rangle = |\psi_0\rangle + \frac{1}{E-\hat{H}}\hat{V}|\psi_0\rangle$

2: $|\psi\rangle = |\psi_0\rangle + \frac{1}{E-\hat{H}}\hat{V}|\psi_0\rangle + \frac{1}{E-\hat{H}}\hat{V}\frac{1}{E-\hat{H}}\hat{V}|\psi_0\rangle + \dots$

Back to the Schrodinger equation

$$(\hat{H}_0 + \hat{V})|\psi\rangle = E|\psi\rangle$$

renormalize $\hat{V} = \frac{\hbar}{2m} U(\vec{r})$; $E = \frac{\hbar^2 k^2}{2m}$ $|\psi_0\rangle = |k\rangle$

$$(\nabla^2 + k^2)\,\psi(\vec{r}) = U(\vec{r})\,\psi(\vec{r})$$ $\psi_0(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikz}$

Standart solution method : Green's function

Assume $(\nabla^2 + k^2)\, G(\vec{r}, \vec{r}\,') = \delta^{(3)}(\vec{r} - \vec{r}\,')$

If $\psi_0(\vec{r})$ satisfies the homogeneous eqn

$$(\nabla^2 + k^2)\,\psi_0(\vec{r}) = 0,$$

then

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \int G(\vec{r} - \vec{r}\,')\, U(\vec{r}\,')\, \psi(\vec{r}\,')\, dV$$

Check

$$(\nabla^2 + k^2)\psi(\vec{r}) = \overbrace{(\nabla^2 + k^2)\psi_0(\vec{r})}^{=0} + \int \overbrace{(\nabla_{\vec{r}}^2 + k^2) G(\vec{r}, \vec{r}\,')}^{\delta^{(3)}(\vec{r} - \vec{r}\,')}\, U(\vec{r}\,')\, \psi(\vec{r}\,')\, dV$$

$$= U(\vec{r})\,\psi(\vec{r})$$

Green's function $G_\pm(\vec{r}, \vec{r}\,') = -\frac{1}{4\pi} \frac{e^{\pm ik|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}$

(for derivations - SN section 6.2)

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikz} - \frac{1}{4\pi} \frac{2m}{\hbar^2} \int \frac{e^{\pm ik|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|} V(\vec{r}\,')\, \psi(\vec{r}\,')$$

clearly G_+ provides an outgoing spherical wave for $\vec{r} \gg \vec{r}\,'$ (assymptotics)

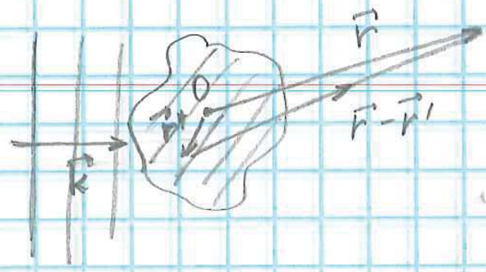
Since we are looking for a solution

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \left(e^{ikz} + f(\theta, \varphi)\, e^{ikr}/r \right)$$

we should be using $G_+(\vec{r}, \vec{r}\,') = -\frac{1}{4\pi} \frac{e^{ik|\vec{r} - \vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|}$

Finite-range scattering potential

$$\hat{r} = \frac{\vec{r}}{r}$$



$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikz} - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\,\psi(\vec{r}')\,dV'$$

for $|\vec{r}| >> |\vec{r}'|$ within the scattaring potential range

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + \cancel{\vec{r}'^2}} \approx r\sqrt{1 - 2\hat{r}\cdot\frac{\vec{r}'}{r}} \approx r - \hat{r}\cdot\vec{r}'$$

$$\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \frac{e^{ikr}}{r}\, e^{-i\underbrace{k\hat{r}}_{\vec{k}'}\vec{r}'} = \frac{e^{ikr}}{r}\, e^{-i\vec{k}'\vec{r}'}$$

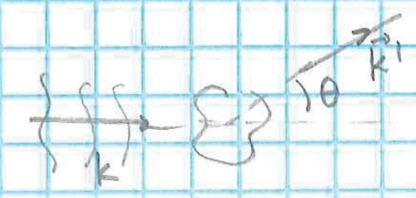
$$(*)\quad \psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}'\vec{r}'} V(\vec{r}')\,\psi(\vec{r}')\,dV'$$

$$= \frac{1}{(2\pi\hbar)^{3/2}}\left[e^{ikz} - \frac{m}{2\pi\hbar^2}(2\pi\hbar)^{3/2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}'\vec{r}'} V(\vec{r}')\,\psi(\vec{r}')\,dV' \right]$$

$$f(\theta,\varphi) \equiv f(\vec{k},\vec{k}')$$

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}(2\pi\hbar)^{3/2}\int e^{-i\vec{k}'\vec{r}'} V(\vec{r}')\,\psi(\vec{r}')\,dV'$$

(*) is not a true solution, but an integral equation $\rightarrow$ difficult to solve



Born approximation

Iterative procedure

1st step! use $|\psi\rangle = |\psi_0\rangle = |k\rangle$ on RHS

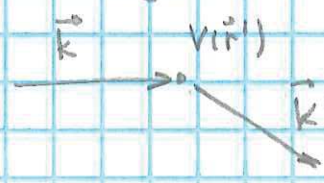
$$f^{(1)}(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\cancel{(2\pi\hbar)^{3/2}} \int e^{i\vec{k}'\vec{r}'}\, V(\vec{r}')\, \frac{1}{\cancel{(2\pi\hbar)^{3/2}}}\, e^{i\vec{k}\vec{r}'}\, d^3\vec{r}'$$

$$f^{(1)}(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k}-\vec{k}')\vec{r}'}\, V(\vec{r}')\, dV'$$

and

$$\frac{d\sigma^{(1)}}{d\Omega} = |f^1(\vec{k},\vec{k}')|^2 = \frac{m^2}{4\pi^2\hbar^4}\left|\int e^{i(\vec{k}-\vec{k}')\vec{r}'} V(\vec{r}')\, dV'\right|^2$$

First order approximation accounts for a single scattering event



Slow particles $ka \ll 1$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int V(\vec{r}')\,dV'\right|^2$$ isotropic

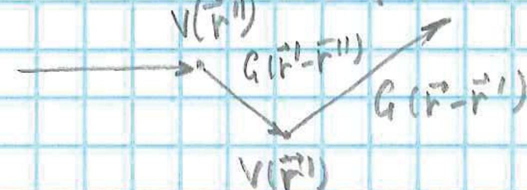
Fast particles $ka \gg 1$

most scattering is for $\theta \approx 0$

Higher-order approximation

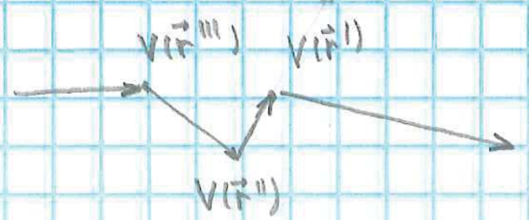
$$\psi(\vec{r}) = \psi_0(\vec{r}) + \underbrace{\int dV'\; G(\vec{r}-\vec{r}')V(\vec{r}')\psi_0(\vec{r}')}_{\Delta\psi^{(1)}(\vec{r})} +$$

$$+ \underbrace{\int dV'dV''\; G(\vec{r}-\vec{r}')V(\vec{r}')G(\vec{r}'-\vec{r}'')V(\vec{r}'')\psi_0(\vec{r}'')}_{\Delta\psi^{(2)}(\vec{r})} +$$



second order ⟹ Two scattering events

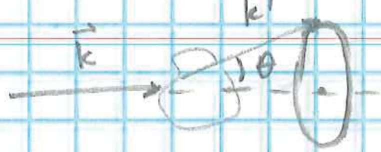
$+$



Third order ⟹ Three scattering events

$$f^{(1)}(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int dV' \, e^{i(\vec{k}-\vec{k}')\vec{r}'}\, V(\vec{r}')$$

For a spherically symmetric potential $V(\vec{r}) = V(r)$
the solution must be cyllindrically symmetric



$$\vec{k}-\vec{k}' = \vec{q}$$

$$|\vec{q}| = q = |\vec{k}-\vec{k}'| = \sqrt{k^2+2k^2\cos\theta+k^2}$$
$$= 2k\sin\theta/2$$

$$f^{(1)}(\theta) = -\frac{m}{2\pi\hbar^2}\cdot 2\pi \int \sin\theta' d\theta'\, r'^2 dr'\, V(r')\, e^{i\vec{q}\vec{r}'} = \qquad (= iqr\cos\theta)$$

$$= -\frac{m}{\hbar^2}\frac{1}{iq}\int_0^\infty \frac{r'^2}{r'} V(r')\left(e^{iqr'} - e^{-iqr'}\right) dr' =$$

$$= -\frac{2m}{\hbar^2 q}\int_0^\infty r' V(r')\sin qr'\, da$$

Example: a solid sphere

$$V(\vec{r}) = \begin{cases} V_0 & r \le a \\ 0 & r > a \end{cases}$$

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 q}\int_0^a r'\sin qr'\, dr' = -\frac{2mV_0}{\hbar^2 q^3}\underbrace{\int_0^{qa} x\sin x\, dx}_{-x\cos x + \sin x} =$$

$$= -\frac{2m}{\hbar^2}\frac{V_0 a_0^3}{(qa)^2}\left[\frac{\sin qa}{qa} - \cos qa\right]$$

Solutions $\sin qa = (qa)\cos qa$ $\qquad qa = 4.49;\ 7.73;\ 10.9 \ldots$

For each of these point $f^{(1)} = 0$