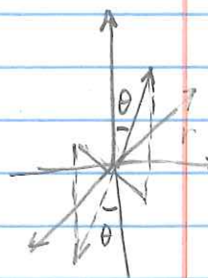
Discrete symmetries

Spatial inversion symmetry or Parity Symmetry

Parity operation $\equiv$ space inversion



$\vec{r} \longrightarrow -\vec{r}$ parity odd vector

$\vec{p} = m\frac{dr}{dt} \longrightarrow -\vec{p}$ parity odd vector

angular momentum

$\vec{L} = \vec{r} \times \vec{p} \longrightarrow \vec{L}$ parity even pseudovector

kinetic energy

$K = \frac{1}{2} m v^2 \longrightarrow K$ parity even scalar

Electric field parity odd vector

$\vec{E} = \int \frac{\rho(\vec{r})\,\vec{r}}{r^3} dV$

Magnetic field parity even pseudovector

$\vec{B} = \frac{1}{c} \int \frac{\vec{j} \times \vec{r}}{r^3} dV$

Standard notation

Vector (or polar vector) : transforms as vector under rotation, odd parity

Pseudovector (or axial vector): — " — vector — " —, even parity

Scalar — " — scalar, even

Pseudoscalar — " — scalar, odd

Quantum mechanical parity operator Π

If we space invert the state, then the expectation value of the position must be inverted

$$(\langle \alpha|\Pi^+)\vec{r}(\Pi|\alpha\rangle) = -\langle \alpha|\vec{r}|\alpha\rangle$$

To be valid for any state

$$\Pi^+ \vec{r}\, \Pi = -\vec{r}$$

Also $\Pi^+\Pi = \Pi\Pi^+ = 1 \Rightarrow \Pi$ is unitary operator

$\vec{r}\,\Pi = -\Pi\vec{r}$ — $\vec{r}$ & Π anticommute

In the basis of eigenfunctions of the position operator (x- representation)

$\Pi|\vec r\rangle$ – ?

$\hat{\vec r}\,\Pi|\vec r'\rangle = -\Pi\hat{\vec r}|\vec r'\rangle = -\vec r'\,\Pi|\vec r'\rangle$

$\vec r'$ is the eigen value of $|\vec r'\rangle$

$-\vec r'$ is the —"— of $\Pi|\vec r'\rangle$

thus

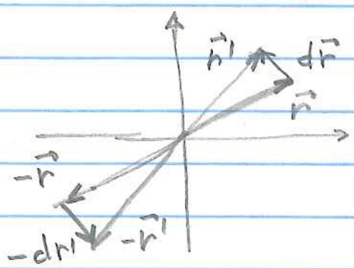
$\Pi|\vec r'\rangle = \cancel{e^{i\delta}}|-\vec r'\rangle \Rightarrow \Pi|\vec r'\rangle = |-\vec r'\rangle$

by convention

$\Pi^2|\vec r\rangle = |\vec r\rangle \Rightarrow \Pi^{-1} = \Pi$

eigenvalues of $\Pi = \pm 1$ (even/odd parity)

Momentum:



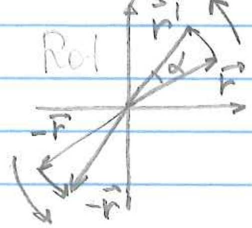
Translation followed by parity = = parity followed by translation in <u>opposite</u> direction

$$\Pi\, T(d\vec r) = T(-d\vec r)\,\Pi$$

$$\Pi\left(1 - \frac{i\vec p\, d\vec r}{\hbar}\right) = \left(1 + \frac{i\vec p\, d\vec r}{\hbar}\right)\Pi$$

$$-\Pi\vec p = \vec p\,\Pi$$ – anti commute

Angular momentum ($\vec L, \vec S, \vec J$, etc)



Rotation followed by parity = = parity followed by rotation

$$\Pi\vec L = \vec L\,\Pi$$ – commute

pseudo vectors

Wavefunctions under parity

State $|\alpha\rangle$, wavefunction $\psi_\alpha(\vec{r}) = \langle \vec{r}|\alpha\rangle$

$\Pi|\vec{r}\rangle = |-\vec{r}\rangle$

$\Pi|\alpha\rangle \Rightarrow \langle \vec{r}|\Pi|\alpha\rangle = \langle -\vec{r}|\alpha\rangle = \psi_\alpha(-\vec{r})$

Not every wavefunction is an eigenfunction of Π but assume it is

$\Pi|\alpha\rangle = \pm|\alpha\rangle$ (even/odd)

$\psi(-\vec{r}) = \pm\psi(\vec{r})$

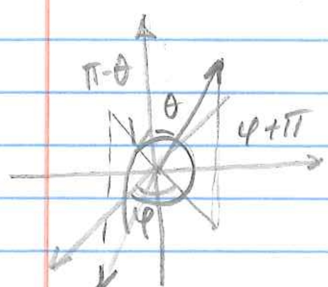
Examples: plane wave $\psi(\vec{r}) = e^{i\vec{k}\vec{r}}$ [have clear direction]

$\psi(-\vec{r}) \neq \pm\psi(\vec{r})$

Spherical functions $[\Pi, \vec{L}] = 0$

we can find a common set of eigenfunctions

$\Pi\,|r, \theta, \varphi\rangle = |r, \pi-\theta, \varphi+\pi\rangle$



$Y_{\ell m} \propto P_\ell^m(\cos\theta)\, e^{im\varphi}$

$\Pi\, Y_{\ell m} = Y_{\ell m}(\pi-\theta, \varphi+\pi) = (-1)^\ell\, Y_{\ell m}$

$\Pi|\_\_, \ell m\rangle = (-1)^\ell |\_\_, \ell, m\rangle$

Orbital momentum quantum number determines the parity of the state

Selection rules

$\Pi|\alpha\rangle = \lambda_\alpha|\alpha\rangle$ $\lambda_\alpha = \pm 1$

$\Pi|\beta\rangle = \lambda_\beta|\beta\rangle$

Then $\langle\alpha|\hat{A}|\beta\rangle = \langle\alpha|\Pi\cdot\Pi\hat{A}\,\Pi\cdot\Pi|\beta\rangle = \lambda_\alpha\lambda_\beta\langle\alpha|\Pi\hat{A}\Pi|\beta\rangle$

If the operator $\hat{A}$ has defined parity

$\Pi\hat{A} = \lambda_A\,\hat{A}\Pi$ then

$\langle\alpha|\hat{A}|\beta\rangle = \lambda_\alpha\lambda_\beta\lambda_A\,\langle\alpha|\hat{A}|\beta\rangle$

So unless $\lambda_\alpha \cdot \lambda_\beta \cdot \lambda_A = 1$, $\langle \alpha | \hat{A} | \beta \rangle = 0$

For example, electro-dipole transitions b/w atomic levels

$$\langle \alpha | z | \beta \rangle = - \lambda_\alpha \lambda_\beta \langle \alpha | z | \beta \rangle$$

so states $|\alpha\rangle$ and $|\beta\rangle$ must be of different parity

We already used this many times, for example for stark effect calculations (only s & p states are mixed by an electric field)

If Hamiltonian commutes with parity operator $[\hat{H}, \pi] = 0$ then its eigenstates are also parity eigenvectors (i.e have distinct parity), at least for a degenerate spectrum

Hydrogen hamiltonian

$$\hat{H} = \frac{p^2}{2m} - \frac{e^2}{r} - \frac{p^4}{8m^3c^2} + \frac{1}{2}\frac{e^2}{m^2c^2}\frac{1}{r^3}\vec{L}\cdot\vec{S} + \frac{A}{\hbar^2}\vec{I}\cdot\vec{J}$$

$$- \frac{e}{2mc}(\vec{L} + 2\vec{S})\vec{B} - e\vec{r}\vec{E}$$

all terms are even, so all processes are parity-invariant.