Charge conjugation

Electron: $(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi_e = 0$

Can we figure out what operator transforms ψ_e into ψ_p - positron wavefunction? $\psi_p = U_c \psi_e$

Anti-particle : $e \to -e$

$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m)\psi_p = 0$$

Somewhat similar to complex conjugate of DE

$$(-i(\gamma^\mu)^* \partial_\mu - e(\gamma^\mu)^* A_\mu - m)\psi_e^* = 0$$

What if there exist an operator $\tilde{C}$ such that $\tilde{C}^{-1}\tilde{C} = \tilde{C}\tilde{C}^{-1} = \hat{\mathbb{1}}$, but $\tilde{C}(\gamma^\mu)^*\overbrace{\tilde{C}}^{\hat{\mathbb{1}}} = -\gamma^\mu$

Indeed $\tilde{C}(-i(\gamma^\mu)^* \partial_\mu - e(\gamma^\mu)^* A_\mu - m)\tilde{C}^{-1}\tilde{C}\psi_e^* = 0$

$$(-i\underbrace{\tilde{C}(\gamma^\mu)^*\tilde{C}^{-1}}_{-\gamma^\mu}\partial_\mu - e\underbrace{\tilde{C}(\gamma^\mu)^*\tilde{C}^{-1}}_{-\gamma^\mu}A_\mu - m)\tilde{C}\psi_e^* = 0$$

$$(+i\gamma^\mu\partial_\mu + e\gamma^\mu A_\mu - m)\tilde{C}\psi_e^* = 0 \quad \Rightarrow \quad \psi_p = \tilde{C}\psi_e^*$$

What is $\tilde{C}$? $(\gamma^\mu)^* = \begin{cases} \gamma^\mu & \text{for } \mu = 0,1,3 \\ -\gamma^\mu & \text{for } \mu = 2 \end{cases}$

We **know** that $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} I$ and $\beta\alpha_j + \alpha_j\beta = 0$

$$\alpha_j^{-1}\alpha_i\alpha_j + \alpha_j^{-1}\alpha_j\alpha_i = 2\alpha_i^{-1} \qquad \alpha_j^{-1}\beta\alpha_j + \alpha_j^{-1}\alpha_j\beta = 0$$

$$\alpha_j^{-1}\alpha_i\alpha_j = 2\alpha_i^{-1} - \alpha_i \qquad \alpha_j^{-1}\beta\alpha_j = -\beta \quad \checkmark$$

we can show that the requirement $\tilde{C}^{-1}(\gamma_\mu)^*\tilde{C} = -\gamma_\mu$ works for $\tilde{C} = i\gamma^2$

$$\psi_p = \tilde{C}\psi_e^* = i\gamma^2\psi_e^* = \underbrace{i\gamma^2\gamma^0}_{U_c}(\bar{\psi}_e)^T$$

U_c - charge conjugation operator

$$\psi_e = \begin{pmatrix} u \\ v \end{pmatrix} e^{-ip^\mu x_\mu}$$

$$\psi_p = i\gamma^2 \begin{pmatrix} u \\ v \end{pmatrix}^* e^{ip^\mu x_\mu} = i\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}^* e^{ip^\mu x_\mu} = \begin{pmatrix} i\sigma_y v \\ i\sigma_y u \end{pmatrix} e^{ip^\mu x_\mu}$$

$\vec{r} \to -\vec{r}$, $t \to -t$ (plus switches positive - negative energy parts)

Time reversal (anti-unity operator)

$\Theta = U \cdot K \leftarrow$ complex conjugation

$\uparrow$ unitary

$$i\partial_t \psi = [-i\gamma^0\gamma^i\partial_i + \gamma^0 m]\psi$$
$$t \to -t; \quad \psi \to \mathcal{T}\psi = U_T K\psi = U_T\psi^*$$

$$\mathcal{T}\, i\partial_t\, \mathcal{T}^{-1}\mathcal{T}\psi = \mathcal{T}[-i\gamma^0\gamma^i\partial_i + \gamma^0 m]\mathcal{T}^{-1}\mathcal{T}\psi$$
$$U_T\cdot K\,(i\partial_t)K U_T^{-1}U_T K\psi = U_T K[-i\gamma^0\gamma^i\partial_i + \gamma^0 m]K U_T^{-1}U_T K\psi$$
$$\underset{\overset{\shortparallel}{i\partial_{t'}(U_T\psi^*)}}{-i\partial_t U_T\psi^*} = [iU_T K\gamma^0\gamma^i\partial_i K U_T + U_T\gamma^0 U_T^{-1}m]\,U_T\psi^*$$

For $(U_T\psi^*)$ to be the solution of DE, we need

$$U_T K\gamma_0\gamma^i K U_T^{-1} = -\gamma^0\gamma^i \qquad U_T\gamma^0 U_T^{-1} = \gamma_0$$
$$U_T = \gamma^1\gamma^3 \qquad \mathcal{T} = \gamma^1\gamma^3 K$$

Parity $\quad x^i \to -x^i \qquad \psi \to \mathcal{P}\psi$

$$i\partial_t(\mathcal{P}\psi) = [-i\,\mathcal{P}\gamma^0\gamma^i\mathcal{P}^{-1}(-\partial_i) + \mathcal{P}\gamma_0\mathcal{P}^{-1}m]\mathcal{P}\psi$$

$$\mathcal{P}\gamma^0\gamma^i\mathcal{P}^{-1} = -\gamma^0\gamma^i \qquad \mathcal{P} = \gamma^0\,\pi$$
$$\mathcal{P}\gamma_0\mathcal{P} = \gamma^0$$

$\uparrow$ parity operator

CPT symmetry

$$CP\mathcal{T}\;\psi(r,t) = i\gamma^2[\mathcal{P}\mathcal{T}\psi(r,t)]^* = i\gamma^2\gamma^0[\mathcal{T}\psi(-r,t)]^* =$$
$$= i\gamma^2\gamma^0\gamma^1\gamma^3[K\psi(-r,t)]^* = i\gamma^2\gamma^0\gamma^1\gamma^3\,\psi(-\bar{r},t) =$$
$$= \underbrace{i\gamma^0\gamma^1\gamma^2\gamma^3}_{\gamma^5}\,\psi(-\bar{r},t)$$

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

$\{\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^5\}$ and their combinations form a complete basis of 4×4 matrices

$CP\mathcal{T}\,\psi_{particle} = \psi_{antiparticle}$ total symmetry b/w matter and antimatter

Hydrogen atom – Dirac style

Dirac Hamiltonian $\hat{H}_D = \vec{\alpha}\vec{p} + \beta m - \frac{Ze^2}{r}$

exact covariant form, $\vec{A} = 0$

$[\hat{H}_D, \hat{\mathcal{P}}] = 0 \rightarrow$ energy eigenstates have distinct parity

$[\hat{H}_D, \hat{\vec{J}}] = 0 \longrightarrow$ energy eigenstates are also the eigenstates of $\hat{J}^2, \hat{J}_z$

$$\left[\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\vec{p} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(V-E)\right]\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0$$

$$\begin{pmatrix} V-E+m & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & V-E-m \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0$$

changes sign under parity

$$\chi = -\frac{1}{V-E-m}\vec{\sigma}\vec{p}\varphi$$

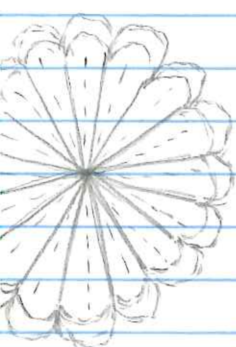
χ, φ – different parity

$$J^2\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = J(J+1)\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$J_3\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = M\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$J \pm 1/2$

$$\mathcal{P}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = (-1)^L\begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$



Angular part of the solution $\mathcal{Y}^{jm}_{j\pm 1/2}$ spinors

We are looking for a solution in form

$$\varphi = \frac{F(r)}{r}\mathcal{Y}^{jm}_{j+1/2} \quad \text{or} \quad \varphi = \frac{F(r)}{r}\mathcal{Y}^{jm}_{j-1/2}$$

$$\chi = i\frac{G(r)}{r}\mathcal{Y}^{jm}_{j-1/2} \qquad \chi = i\frac{G(r)}{r}\mathcal{Y}^{jm}_{j+1/2}$$

We need to rewrite the Dirac equ in a spherically symmetric form. The main "problematic" term is "$\vec{\alpha}\cdot\vec{p}$" term.

We need to show that we can rewrite it

$$(\vec{\alpha}\cdot\vec{r})(\vec{\alpha}\cdot\vec{p}) = \begin{pmatrix} \vec{\sigma}\cdot\vec{r}\,\vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{r}\,\vec{\sigma}\cdot\vec{p} \end{pmatrix} = (\vec{\sigma}\vec{r})(\vec{\sigma}\vec{p})\,\hat{1} =$$

$$= [\vec{r}\cdot\vec{p} + i\vec{\sigma}\cdot\vec{r}\times\vec{p}]\,\hat{1} = [r p_r + i(1+\vec{\sigma}\vec{L})]\cdot\hat{1} =$$

$$= \left[-i\frac{\partial}{\partial r}r + i(1+\vec{\sigma}\vec{L})\right]\hat{1}$$

but at the same time

$$(\vec{\alpha}\vec{r})(\vec{\alpha}\vec{r})(\vec{\alpha}\vec{p}) = (\vec{\alpha}\vec{r})^2(\vec{\alpha}\vec{p}) = r^2(\vec{\alpha}\vec{p})$$

$$\vec{\alpha}\vec{p} = \alpha_r\left[\frac{1}{i}\frac{\partial}{r\,\partial r} + \frac{i}{r}(1+\vec{\sigma}\vec{L})\right] \quad \text{where } \alpha_r = \frac{\vec{\alpha}\vec{r}}{r}$$

$$(1+\vec{\sigma}\vec{L})\hat{1} = \hat{1} + \begin{pmatrix} \vec{\sigma}\vec{L} & 0 \\ 0 & \vec{\sigma}\vec{L} \end{pmatrix} = \hat{1} + \begin{pmatrix} 2\vec{S}\vec{L} & 0 \\ 0 & 2\vec{S}\vec{L} \end{pmatrix} =$$

$$= (1 + J^2 - L^2 - \underbrace{S^2}_{3/4})\,\hat{1} = (J^2 + L^2 + 1/4)\,\hat{1} \quad \text{for eigen spin states}$$

Almost there — need to figure out $\hat{L}^2$!

$$L^2 \mathcal{Y}^{jm}_{j+1/2} = \ell(\ell+1)\,\mathcal{Y}^{jm}_{j+1/2} = (j+1/2)(j+1+1/2)\,\mathcal{Y}^{jm}_{j+1/2}$$

$$L^2 \mathcal{Y}^{jm}_{j-1/2} = (j-1/2)(j+1-1/2)\,\mathcal{Y}^{jm}_{j-1/2}$$

$\varphi \propto \mathcal{Y}^{jm}_{j+1/2}$

$\chi \propto \mathcal{Y}^{jm}_{j-1/2}$

$$L^2\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = (J+\beta\cdot 1/2)(J+1+\beta 1/2)\begin{pmatrix}\varphi\\ \chi\end{pmatrix} =$$

$$= \left[J(J+1) + \frac{1}{4} + \frac{1}{2}(2J+1)\beta\right]\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

or $-1/2$ for the other combination

$$(1+\vec{\sigma}\vec{L}) = -\frac{\varepsilon}{2}(2J+1)\beta \quad \text{where} \quad \varepsilon = +1 \text{ for } \begin{pmatrix} \mathcal{Y}^{jm}_{j+1/2} \\ \mathcal{Y}^{jm}_{j-1/2} \end{pmatrix}$$

$$\text{and} \quad \varepsilon = -1 \text{ for } \begin{pmatrix} \mathcal{Y}^{jm}_{j-1/2} \\ \mathcal{Y}^{jm}_{j+1/2} \end{pmatrix}$$

So the radial form of the Dirac eqn is

$$\left[\alpha_r\left(\frac{1}{ir}\frac{\partial}{\partial r}r - \frac{i\varepsilon}{r}(J+\tfrac{1}{2})\beta\right) + m\beta + V - E\right]\psi^{jm}_{\varepsilon}$$

where $$\psi^{jm}_{\varepsilon} = \begin{pmatrix} F(r)/r & \mathcal{Y}^{jm}_{j+\varepsilon/2} \\ iG(r)/r & \mathcal{Y}^{jm}_{j-\varepsilon/2} \end{pmatrix}$$

$$\alpha_r = \frac{\vec{\alpha}\vec{r}}{r} = \begin{pmatrix} 0 & \frac{\vec{\sigma}\vec{r}}{r} \\ \frac{\vec{\sigma}\vec{r}}{r} & 0 \end{pmatrix}$$

Parity

$$\Pi\left[\frac{\vec{\sigma}\vec{r}}{r}\mathcal{Y}^{jm}_{j\pm 1/2}\right] = -\frac{\vec{\sigma}\vec{r}}{r}\left[\Pi\,\mathcal{Y}^{jm}_{j\pm 1/2}\right] = -\frac{\vec{\sigma}\vec{r}}{r}\left((-1)^{j\pm 1/2}\mathcal{Y}^{jm}_{j\pm 1/2}\right)$$

So $\left[\frac{\vec{\sigma}\vec{r}}{r}\mathcal{Y}^{jm}_{j\pm 1/2}\right]$ is an eigenfunction of Π

with eigenvalues $(-1)^{j\mp 1/2}$

Thus $$\frac{\vec{\sigma}\vec{r}}{r}\mathcal{Y}^{jm}_{j\pm 1/2} = \text{Const}\cdot\mathcal{Y}^{jm}_{j\mp 1/2}$$

$$\left(\frac{\vec{\sigma}\vec{r}}{r}\right)^2 = \frac{\vec{r}\cdot\vec{r}}{r^2} + \cancel{i\vec{\sigma}\,\overbrace{\frac{\vec{r}\times\vec{r}}{r^2}}^{=0}} = 1$$

we can show that $\mathcal{Y}^{jm}_{j\pm 1/2} = -\mathcal{Y}^{jm}_{j\mp 1/2}$

$$\alpha_r \psi_\varepsilon^{jm} = \begin{pmatrix} 0 & \frac{\vec{\sigma}\vec{r}}{r} \\ \frac{\vec{\sigma}\vec{r}}{r} & 0 \end{pmatrix} \begin{pmatrix} \frac{F}{r} \mathcal{Y}^{jm}_{j+\varepsilon/2} \\ \frac{iG}{r} \mathcal{Y}^{jm}_{j-\varepsilon/2} \end{pmatrix} = \begin{pmatrix} -\frac{iG}{r} \mathcal{Y}^{jm}_{j+\varepsilon/2} \\ -\frac{F}{r} \mathcal{Y}^{jm}_{j-\varepsilon/2} \end{pmatrix}$$

$$\beta \psi_\varepsilon^{jm} = \begin{pmatrix} F/r\, \mathcal{Y}^{jm}_{j+\varepsilon/2} \\ -\frac{iG}{r} \mathcal{Y}^{jm}_{j-\varepsilon/2} \end{pmatrix}$$

$$\alpha_r \beta \psi_\varepsilon^{jm} = \begin{pmatrix} \frac{iG}{r} \mathcal{Y}^{jm}_{\varepsilon+1/2} \\ -\frac{F}{r} \mathcal{Y}^{jm}_{\varepsilon-1/2} \end{pmatrix}$$

Combining all these expression and separating two spinors, we can get the following equations

$$[E-m-V(r)]F + \left[\frac{d}{dr} - \frac{\varepsilon}{r}(J+\tfrac{1}{2})\right]G = 0$$

$$[E+m-V(r)]G - \left[\frac{d}{dr} + \frac{\varepsilon}{r}(J+\tfrac{1}{2})\right]F = 0$$

Boundary conditions ψ_ε^{jm} is finite at $r=0$

$F(0) = G(0) = 0$

For $r \to \infty$ $\quad V(r) \propto 1/r = 0 \quad \frac{\varepsilon}{r} \to 0$

$\langle V \rangle < 0$

$|E| < |m|$

$$\begin{cases} (E-m)F + \frac{dG}{dr} = 0 \\ (E+m)G - \frac{dF}{dr} = 0 \end{cases} \Rightarrow \quad \frac{dG}{dr} = (m-E)F \qquad \frac{d^2F}{dr^2} = (m^2-E^2)F = K^2 F$$

$\varkappa = \sqrt{m^2 - E^2}$

$$G = \frac{1}{E+M}\frac{dF}{dr} = -a\sqrt{\frac{m-E}{m+E}}\, e^{-\varkappa r} \qquad F = a e^{-\varkappa r}$$

assymtoric behavior

$$\psi_\varepsilon^{jm} \to a \begin{pmatrix} \frac{1}{r} e^{-\varkappa r} \mathcal{Y}^{jm}_{j\pm\varepsilon/2} \\ -\sqrt{\frac{m-E}{m+E}}\, \frac{1}{r} e^{-\varkappa r} \mathcal{Y}^{jm}_{j\pm\varepsilon/2} \end{pmatrix} \quad \text{for } r \to \infty$$

Hydrogen atom solution

$$\varkappa = \sqrt{m^2 - E^2} \qquad \nu = \sqrt{\frac{m-E}{m+E}} \qquad \zeta = Ze^2 \qquad \tau = \varepsilon\left(J + \tfrac{1}{2}\right)$$

$$\rho = \varkappa r$$

$$\begin{cases} \left[\dfrac{d}{d\rho} - \dfrac{\tau}{\rho}\right] G = \left[-\nu + \dfrac{\zeta}{\rho}\right] F \\ -\left[\dfrac{d}{d\rho} + \dfrac{\tau}{\rho}\right] F = \left[\dfrac{1}{\nu} + \dfrac{\zeta}{\rho}\right] G \end{cases}$$

Similar to the non-relativistic case, we are going to look for a solution in a form

$$F(\rho) = \rho^s e^{-\rho} \sum_{k=0}^{\infty} a_k \rho^k$$

$$G(\rho) = \rho^s e^{-\rho} \sum_{k=0}^{\infty} b_k \rho^k$$

$$\frac{dF}{d\rho} = \rho^s e^{-\rho}\left[s a_0 \frac{1}{\rho} + \sum_{k=0}^{\infty}\{(s+k+1)a_{k+1} - a_k\}\rho^k\right]$$

$$\frac{dG}{d\rho} = \rho^s e^{-\rho}\left[s b_0 \frac{1}{\rho} + \sum_{k=0}^{\infty}\{(s+k+1)b_{k+1} - b_k\}\rho^k\right]$$

$$\frac{1}{\rho}F = \rho^s e^{-\rho}\left[a_0 \frac{1}{\rho} + \sum_{k=0}^{\infty} a_{k+1}\rho^k\right]$$

$$\frac{1}{\rho}G = \rho^s e^{-\rho}\left[b_0 \frac{1}{\rho} + \sum_{k=0}^{\infty} b_{k+1}\rho^k\right]$$

$$\left(s b_0 \frac{1}{\rho} - \frac{\tau}{\rho} b_0\right) + \sum_{k=0}^{\infty}\left\{(s+k+1)b_{k+1} - b_k - \frac{b_{k+1}}{\tau}\right\}\rho^k =$$

$$= \frac{\zeta a_0}{\rho} + \sum_{k=0}^{\infty}\left[-\nu a_k + \zeta a_{k+1}\right]\rho^k$$

$$\left(-s a_0 \frac{1}{\rho} - \frac{\tau}{\rho} a_0\right) + \sum_{k=0}^{\infty}\left\{-(s+k+1)a_{k+1} + a_k - \frac{a_{k+1}}{\tau}\right\}\rho^k =$$

$$= \frac{\zeta b_0}{\rho} + \sum_{k=0}^{\infty}\left\{\frac{1}{\nu} a_k + \zeta a_{k+1}\right\}\rho^k$$

$$\begin{cases} (s+k+1-\tau)\, b_{k+1} - b_k + \nu a_k - \xi a_{k+1} = 0 \\ -(s+k+1+\tau)\, a_{k+1} + a_k - \frac{1}{\nu} b_k + \xi b_{k+1} = 0 \end{cases}$$

true for all a_k, b_k, if $a_{-1}, b_{-1} = 0$

$$(s+k+1-\tau+\nu\xi)\, b_{k+1} + [\nu(s+k+1+\tau) - \xi]\, a_{k+1} = 0$$

We are interested in a finite polynomial solution $a_{k'}, b_{k'} = 0$ for $\forall k' \geq k+1$

$$b_{k'} = \nu a_{k'} \quad , \quad k' = k+1$$

$$\nu(s+k'-\tau+\nu\xi) + \nu(s+k'+\tau) - \xi = 0$$

$$2\nu(s+k') + \nu^2\xi - \xi = 0$$

$$2(s+k') = (\nu - 1/\nu)\,\xi$$

$$2(s+k') = \left(\sqrt{\frac{m-E}{m+E}} - \sqrt{\frac{m+E}{m-E}}\right)\xi = \frac{2E}{\sqrt{m^2-E^2}}\,\xi$$

equation for eigen energy spectrum

$$(s+k')^2 = \frac{E^2}{m^2-E^2}\,\xi^2 \Rightarrow E = \frac{m}{\sqrt{1+\frac{\xi^2}{(s+k')^2}}}$$

To determine s, let's look at $k=-1$

$$\begin{cases} (s-\tau)\, b_0 - \xi a_0 = 0 \\ -(s+\tau)\, a_0 - \xi b_0 = 0 \end{cases} \Rightarrow s = \pm\sqrt{\tau^2-\xi^2}$$

Since the wavefunctions scale as $F, G \propto \rho^{s} e^{-\rho}$, in order for them to be linear in the origin, we need to pick only $s>0$

$$s = \sqrt{\tau^2-\xi^2}$$

$$E = \frac{m}{\sqrt{1+\frac{\xi^2}{(k'+\sqrt{\tau^2-\xi^2})}}}$$

$k' = n - (J + 1/2)$ $\quad n = 1, 2, 3, \ldots$

principle quantum number

$$E_{nj} = \frac{m}{\sqrt{1 + \frac{(Ze^2)^2}{\left(n - (J+1/2) + \sqrt{(J+1/2)^2 - (Ze^2)^2}\right)^2}}}$$

or, coming back to normal units

$\left(\frac{e^2}{\hbar c} = \alpha\right)$

$$E_{nj} = \frac{mc^2}{\sqrt{1 + \frac{(Z\alpha)^2}{\left(n - (J+1/2) + \sqrt{(J+1/2)^2 - (Z\alpha)^2}\right)^2}}}$$

Decomposing in orders of α

$$E_{nj} \simeq mc^2 \left[1 - \frac{(Ze^2)^2}{2n^2} - \frac{(Z\alpha^2)^2}{2n^2} \left(\frac{n}{J+1/2} - \frac{3}{4} \right) + \ldots \right]$$

Non-relativistic E_n $\qquad$ first relativistic correction

2 solutions for each E_{nj} - spin degeneracy