Dirac eqn (cont)

Hamiltonian form of the Dirac eqn

$$i\frac{\partial\psi}{\partial t} = \hat{H}\psi \qquad \left(i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi\right)$$

$$\hat{H} = \vec{\alpha}\hat{\vec{p}} + \beta m = \left(-i\hbar c\alpha\nabla + mc^2\beta\right)$$

Covariant form of the Dirac eqn

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\alpha_i = \gamma^0\gamma^i \qquad \beta = \gamma^0$$

Free-particle plane-wave solution

$$\hat{H} = \begin{pmatrix} m & -i\vec{\sigma}\cdot\vec{p} \\ -i\vec{\sigma}\vec{p} & -m \end{pmatrix} \qquad \psi = \begin{pmatrix} u \\ v \end{pmatrix} e^{i\vec{p}\vec{x} - iEt}$$

u, v — 2-component spinors

$$(E-m)u = \vec{\sigma}\vec{p}\,v$$
$$(E+m)v = \vec{\sigma}\vec{p}\,u$$

If $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad v_1 = \frac{1}{E+m}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$ $\quad E > 0$ two independent solutions $p \to 0$

$u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad v_2 = \frac{1}{E+m}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$

If $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad u_1 = \frac{1}{E-m}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$ $\quad E < 0$ two independent solutions $p \to 0$

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If $E_p = \sqrt{E^2 - m^2} > 0$

$E > 0 \quad \psi_{I,II} \propto \sqrt{\frac{m}{E_p}}\, a_{1,2}\, e^{i\bar{p}\bar{x} - iE_p t}$

$E < 0 \quad \psi_{III,IV} \propto \sqrt{\frac{m}{E_p}}\, a_{3,4}\, e^{i\bar{p}\bar{x} + iE_p t}$

$$a_{1,2} = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} \chi_\pm \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p + m}\chi_\pm \end{pmatrix} \qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$a_{3,4} = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} -\frac{\vec{\sigma}\vec{p}}{E_p + m}\chi_\pm \\ \chi_\pm \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\{\psi_I\}$ functions are mutually orthogonal

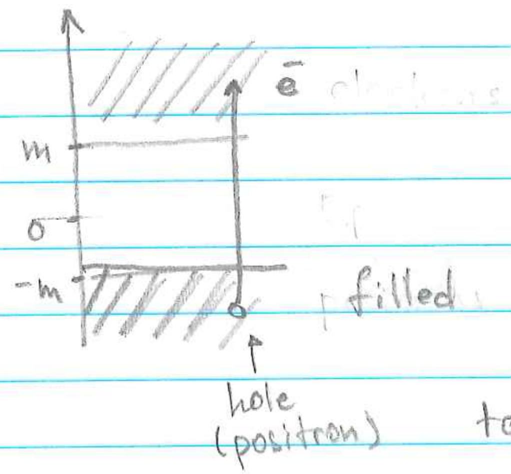
Interpretation: $\psi_{I,II}$ describes electrons with positive energies and spin $\pm 1/2$
$\psi_{III,IV}$ describes "electron" with negative energy (mass) and spin $\pm 1/2$

For $p = 0$

$$\psi_I = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad \psi_{II} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \qquad E > 0$$

$$\psi_{III} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imt} \qquad \psi_{IV} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt} \qquad E < 0$$

Why we don't see negative mass particles (anti-particles)?



All states with $m < 0$ are completely filled $\longrightarrow$ no way for a $m > 0$ $\bar{e}$ to "spontaneously" transition and disappear

However, it may be possible to excite an electron into the positive energy state (with a high-energy photon), leaving a "hole" behind $\longrightarrow$ positron; same as electron but with negative mass and opposite charge. If an $\bar{e}$ and the hole ($\bar{p}$) meet, they are annihilated

Electron spin

Interaction with E-M field

$$p^\mu \to \pi^\mu = p^\mu - eA^\mu$$

$$\hat{H}_{EM} = \vec{\alpha}\vec{\pi} + \beta m$$

Spin in non-relativistic limit $\vec{\mu} = g_s \mu_B \vec{S}$

electron magnetic moment

Non-relativistic limit of Dirac eqn

$$E\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\vec{\pi} \\ +\vec{\sigma}\vec{\pi} & -m \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m & \vec{\sigma}\vec{\pi} \\ \vec{\sigma}\vec{\pi} & -m \end{pmatrix}\begin{pmatrix} u \\ \frac{\vec{\sigma}\vec{\pi}}{E+m} u \end{pmatrix}$$

$$Eu = mu + \frac{(\vec{\sigma}\vec{\pi})(\vec{\sigma}\vec{\pi})}{E+m} u$$

$v \ll c$

$$Ku + \cancel{mu} = \cancel{mu} + \frac{(\vec{\sigma}\vec{\pi})(\vec{\sigma}\vec{\pi})}{2m} u$$

$$\{(\vec{\sigma}\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}(\vec{a}\times\vec{b})\}$$

$$Ku = \frac{\pi^2}{2m} u + i\vec{\sigma}(\vec{\pi}\times\vec{\pi})u$$

$$\vec{\pi}\times\vec{\pi}u = (i\nabla + e\vec{A})\times(i\nabla u + e\vec{A}u) = \cancel{-\nabla\times\nabla u} +$$

$$+ i\nabla\times(eAu) + ieA\times\nabla u + \cancel{e^2\vec{A}\times\vec{A}u} =$$

$$= ie(\nabla\times A)u = ie\vec{B}u$$

$$Ku = \frac{\pi^2}{2m} u - \frac{1}{2m} e\vec{\sigma}\vec{B}\cdot u = \frac{\pi^2}{2m} u - \vec{\mu}\vec{B}\cdot u$$

This is what we call energy in the non-relativistic limit where $\vec{\mu} = \frac{e}{2m}\vec{\sigma} = 2\cdot\frac{e}{2m}\left(\frac{1}{2}\vec{\sigma}\right) = \frac{e}{m}\vec{S} = g_s \mu_B \vec{S}$

$g_s = 2$

Alternatively for $v/c \ll 1$ $\quad H_D \approx m + H_{NR}$

$H_D^2 \approx m^2 + 2m H_{NR}$

$$\hat{H}_D = \vec{\alpha}\vec{\pi} + \beta m$$

$$\hat{H}_D^2 = (\vec{\alpha}\vec{\pi} + \beta m)^2 = (\vec{\alpha}\vec{\pi})(\vec{\alpha}\vec{\pi}) + m^2$$

$$\hat{H}_D^2 - m^2 = \begin{pmatrix} 0 & \vec{\sigma}\vec{\pi} \\ \vec{\sigma}\vec{\pi} & 0 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma}\vec{\pi} \\ \vec{\sigma}\vec{\pi} & 0 \end{pmatrix} = \begin{pmatrix} (\vec{\sigma}\vec{\pi})^2 & 0 \\ 0 & (\vec{\sigma}\vec{\pi})^2 \end{pmatrix} =$$

$$= \begin{pmatrix} \pi^2 - e\vec{\sigma}\vec{B} & 0 \\ 0 & \pi^2 - e\vec{\sigma}\vec{B} \end{pmatrix} = \pi^2 \hat{\mathbb{I}} - e\vec{B}\vec{\Sigma}$$

where $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ is a spin matrix

$$\vec{S} = \frac{1}{2}\vec{\Sigma} \quad \left(\frac{\hbar}{2}\vec{\Sigma}\right)$$

$$\hat{H}_{NR} = \frac{\pi^2}{2m} - \frac{e}{2m}\vec{B}\cdot 2\left(\frac{1}{2}\vec{\Sigma}\right) = \frac{\pi^2}{2m} - \vec{\mu}\vec{B}$$

$$\vec{\mu} = 2\cdot\frac{e}{2m}\cdot\vec{S} = g\mu_B\vec{S}$$

Symmetries of the Dirac Equation

Free particle $H_D = \vec{\alpha}\cdot\vec{p} + \beta m$

+ EM radiation $H_D = \vec{\alpha}(\vec{p} - e\vec{A}) + \beta m + e\Phi$

for $\vec{A} = 0$

$H_D = \vec{\alpha}\vec{p} + \beta m + V_{electric}$

can generalize for any interactions

$$H_D = \vec{\alpha}\vec{p} + \beta m + V(\vec{r}) \quad \text{(no longer covariant)}$$

1. Angular momentum $\vec{L} = \vec{r}\times\vec{p}$

$$[H_D, L_i] = [\vec{\alpha}\cdot\vec{p}, L_i] = [\alpha_\ell p_\ell, \varepsilon_{ijk} x_j p_k] =$$

$$= \varepsilon_{ijk}\alpha_\ell [p_\ell, x_j] p_k \overset{\ell=j}{=} -i\varepsilon_{ijk}\alpha_j p_k = -i[\vec{\alpha}\times\vec{p}]_i \neq 0$$

$\vec{L}$ and H_D do not commute $\Longrightarrow$ no angular momentum conservation

$\vec{S}$ does not commute with $\hat{H}$ either!

$[\beta, \vec{\Sigma}] = 0$ but $[\vec{\alpha}\cdot\vec{p}, \Sigma_i] = [\alpha_k, \Sigma_i] p_k =$

$= 2i\,\varepsilon_{kij}\alpha_i p_k = 2i\,[\vec{\alpha}\times\vec{p}]_i \neq 0$

Total angular momentum $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$

$[H_D, \vec{J}] = 0$ total angular momentum is conserved