Classical electrodynamics

Light is an electromagnetic wave.

We "know" this from Maxwell eqns

no sources

$$\left[\begin{array}{ll} \nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} & \nabla\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t} \\ \nabla\cdot\vec{B} = 0 & \nabla\cdot\vec{D} = 0 \end{array}\right.$$

ε - permittivity
μ - permeability

$\vec{E}$ - electric field

$\vec{D}$ - electric displacement

$\vec{B}$ - magnetic induction

$\vec{H}$ - magnetic field

$$\left.\begin{array}{l}\vec{D} = \varepsilon_0\varepsilon\vec{E} \\ \vec{B} = \mu_0\mu\vec{H}\end{array}\right] \text{isotropic medium}$$

$\varepsilon_0\mu_0 = 1/c^2$

Quantum electrodynamics: $\hat{\vec{E}}, \hat{\vec{D}}, \hat{\vec{B}}, \hat{\vec{H}}$

Each classical field corresponds to an operator for example,

$$\vec{E} = \langle\psi|\hat{\vec{E}}|\psi\rangle$$

Quantum fields must obey Maxwell equations as well! (since they are linear)

$$\nabla\times\langle\psi|\hat{\vec{E}}|\psi\rangle = -\frac{\partial\langle\psi|\hat{\vec{B}}|\psi\rangle}{\partial t}$$

$$\langle\psi|\nabla\times\hat{\vec{E}} + \frac{\partial\hat{\vec{B}}}{\partial t}|\psi\rangle = 0 \qquad \text{for } \forall\psi$$

$$\nabla\times\hat{\vec{E}} + \frac{\partial\hat{\vec{B}}}{\partial t} = 0 \qquad \text{(and so on for other equations)}$$

Similarly $\hat{\vec{D}} = \varepsilon_0\varepsilon\hat{\vec{E}}$, $\hat{\vec{B}} = \mu_0\mu\hat{\vec{H}}$

Classical e-m field energy

$$H = \frac{1}{2}\int dV\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

Corresponding QED Hamiltonian

$$\hat{H} = \frac{1}{2}\int dV\left(\hat{\vec{E}}\cdot\hat{\vec{D}} + \hat{\vec{B}}\cdot\hat{\vec{H}}\right) = \frac{1}{2}\int dV\left(\varepsilon_0\varepsilon\,\hat{\vec{E}}^2(\vec{r},t) + \frac{1}{\mu_0\mu}\hat{\vec{B}}^2(\vec{r},t)\right)$$

$\hat{\vec{E}}$ and $\vec{B}$ are not independent, but connected through Maxwell eqns.

Vector potential $\hat{\vec{A}}$

$\hat{\vec{E}} = -\partial\vec{A}/\partial t \qquad \hat{\vec{B}} = \nabla\times\vec{A}$

Coulumb gauge: $\nabla(\varepsilon\vec{A}) = 0$ [and so $\nabla\vec{D} = 0$].

wave equation for $\hat{\vec{A}}$

$$\frac{1}{\varepsilon}\nabla\times\frac{1}{\mu}\nabla\times\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$$

isotropic medium $\varepsilon \neq \varepsilon(\vec{r}),\ \mu \neq \mu(\vec{r})$

$$\nabla^2\hat{\vec{A}} + \frac{\mu\varepsilon}{c^2}\frac{\partial^2\hat{\vec{A}}}{\partial t^2} = 0$$

Classical problems:

Typical way to proceed: consider boundary conditions for the problem, find a good mode basis that satisfies these boundary conditions $\vec{A}_k(\vec{r},t)$, and then figure out what combinations of these modes provides the right solution

$$\vec{A}(\vec{r},t) = \sum_k c_k \vec{A}_k(\vec{r},t)$$

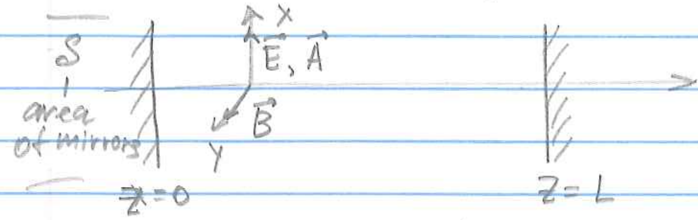
QED solution:

$$\hat{\vec{A}}(\vec{r},t) = \sum_k\left[\vec{A}_k(\vec{r},t)\,\hat{a}_k + \vec{A}_k^*(\vec{r},t)\,\hat{a}_k^+\right]$$

where $\hat{a}_k, \hat{a}_k^+$ are lowering and raising operators for SHO

(usually called annihilation and creation operators in quantum optics)

Simplest case: plane EM wave b/w two perfect conductors



$\varepsilon = 1 \quad \mu = 1$

$\vec{E}(z=0) = \vec{E}(z=L) = 0$ boundary conditions

$$\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t} = 0 \quad \text{or} \quad \nabla^2 \vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t} = 0$$

Spatial modes that automatically satisfy the boundary conditions: standing waves

$\vec{A}, \vec{E} \propto \sin\left[\frac{\pi n z}{L}\right] \qquad n = 1, 2, \ldots$

Single - mode approximation (one particular standing wave is excited)

Let's pick $\vec{E} = (E_x, 0, 0)$, $\vec{B} = (0, B_y, 0)$, $\vec{A} = (A_x, 0, 0)$

$$E_x(\vec{r}, t) = E_x(z, t) = \mathcal{E}(t) \cdot \sin(kz) \qquad k = \frac{\pi n}{L} = \frac{\omega}{c}$$

At that point we are going to "guess" the normalization

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, q(t) \sin kz$$ V - volume of the cavity

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad \frac{\partial B_y}{\partial z} = \frac{1}{c^2}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\, \dot{q}(t) \sin kz$$

$$B_y = \frac{1}{\underset{\omega}{ck}}\sqrt{\frac{2\omega^2}{V\varepsilon_0 c^2}}\, \dot{q}(t)\cos kz = -\sqrt{\frac{2}{V\varepsilon_0 c^2}}\, \dot{q}(t) \cos kz$$

When moving to quantum case

$q \to \hat{q}, \quad \dot{q} \to \hat{p}$ (canonical position and momentum)

Hamiltonian

$$H = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2 + \frac{1}{\mu_0} B_y^2 \right] =$$

$$= \frac{1}{\cancel{2}} \cancel{\varepsilon_0} \left(\frac{\cancel{2}\omega^2}{V \cancel{\varepsilon_0}} \right) q^2(t) \int dV \cdot \sin^2 kz + \frac{1}{\cancel{2}\mu_0} \frac{\cancel{2}}{V \cancel{\varepsilon_0}} (\dot{q}(t))^2 \int dV \cos^2 kz$$

$$\int_V \sin^2 kz \; dz\,dx\,dy = S \int_0^L \sin^2 kz \, dz = S \cdot \frac{L}{2} = \frac{V}{2}$$

$$H = \frac{\omega^2}{\cancel{V}} q^2(t) \cdot \frac{\cancel{V}}{2} + \frac{1}{\cancel{V}} (\dot{q})^2 \frac{\cancel{V}}{2} = \frac{1}{2}\left((\dot{q})^2 + \omega^2 q^2 \right)$$

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2 \right) \qquad \text{SHO !}$$

$$[\hat{q}, \hat{p}] = i\hbar$$

Annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q} + i\hat{p}) \qquad \hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}} (\omega \hat{q} - i\hat{p})$$

$$\left[\begin{array}{l} \hat{E}_x = \sqrt{\dfrac{\hbar\omega}{\varepsilon_0 V}} \, (\hat{a} + \hat{a}^+) \sin kz \\ \hat{B}_y = \dfrac{1}{c}\sqrt{\dfrac{\hbar\omega}{\varepsilon_0 V}} \, (\hat{a} - \hat{a}^+) \cos kz \end{array} \right.$$

$$[a, a^+] = 1 \longrightarrow aa^+ - a^+a = 1$$

$$\hat{H} = \hbar\omega \left(\hat{a}^+\hat{a} + \frac{1}{2} \right)$$

$$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{a}] = i\omega \, [\hat{a}^+\hat{a}, \hat{a}] = i\omega \left(\hat{a}^+\hat{a}\hat{a} - \overbrace{\hat{a}\hat{a}^+}^{1+\hat{a}^+\hat{a}}\hat{a} \right)$$

$$= -i\omega\hat{a}$$

$$\hat{a}(t) = e^{i\omega t}\hat{a}$$

Plane e-m linearly polarized wave inside the cavity

$$\begin{cases} \hat{E}_x = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} + \hat{a}^{+} e^{i\omega t} \right) \sin kz \\ \hat{B}_y = \frac{1}{c}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}} \left(\hat{a} e^{-i\omega t} - \hat{a}^{+} e^{i\omega t} \right) \cos kz \end{cases}$$

Fock states (or number states)

$\hat{n} = \hat{a}^{+}\hat{a}$ $\qquad \hat{n}|n\rangle = n|n\rangle$

$|n\rangle$ - eigenstates of the number operator, describing the state with known number of photons

$|0\rangle$ - vacuum state (no photons)
$|1\rangle$ - single-photon state

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)|n\rangle = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle$$

$$E_n = n\hbar\omega + \frac{\hbar\omega}{2}$$

Why $\hat{a}$ is annihilation operator?

$$\hat{n}(\hat{a}|n\rangle) = \overbrace{\hat{a}^{+}\hat{a}}^{\hat{a}\hat{a}^{+}-1}(\hat{a}|n\rangle) = (\hat{a}\underbrace{\hat{a}^{+}\hat{a}}_{\hat{n}} - \hat{a})|n\rangle =$$

$$= \hat{a}\underbrace{(\hat{n}|n\rangle)}_{=n|n\rangle} - \hat{a}|n\rangle = (n-1)\,\hat{a}|n\rangle \qquad \left[\text{one photon is gone after } \hat{a}\right]$$

$\hat{a}|n\rangle$ is an eigenstate of $\hat{n}$ with eigen value of $(n-1)$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Properties of the number state

$$\langle E_x \rangle = \langle n | E_x | n \rangle = 0$$

$$\Delta E_x = \sqrt{\langle E_x^2 \rangle - \langle E_x \rangle^2} = \sqrt{\langle n | E_x^2 | n \rangle} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\sqrt{2n+1}$$

No well-defined average field, measuremets will give random values; the spread increases with the photon number

Corresponds to the field with undefined phase

Note that even vacuum state $|0\rangle$ has vacuum fluctuations

$$\Delta E_{x\,vac} = \sqrt{\hbar\omega / 2\varepsilon_0 V} > 0$$

Vacuum is alive, it wiggles!

What state would be the closest analogue of the classical e-m wave?

Coherent state

Formally, this state is an eigenstate of the annihilation operator.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Note: $|\alpha\rangle$ is <u>not</u> an eigenstate of $\hat{a}^+$!

$$\langle\alpha|\hat{a}^+ = \alpha\langle\alpha|, \quad \text{but} \quad \hat{a}^+|\alpha\rangle \neq \alpha^*|\alpha\rangle$$

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Properties $\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2$

$$\Delta n = \sqrt{\langle\alpha|\hat{n}^2|\alpha\rangle - \langle\alpha|\hat{n}|\alpha\rangle^2} = |\alpha|$$

Poisson statistics

$$\frac{\Delta n}{n} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{n}}$$

Average value of the electric field

$$\langle\alpha|E_x|\alpha\rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right) \sin kz =$$

$$= 2\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} |\alpha| \cos(\omega t - \varphi) \sin kz$$ – classical description of a standing E-M wave

$$\langle\alpha| E_x^2|\alpha\rangle = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right) \langle\alpha|(\hat{a}e^{-i\omega t} + \hat{a}^+ e^{i\omega t})^2|\alpha\rangle \sin^2 kz.$$

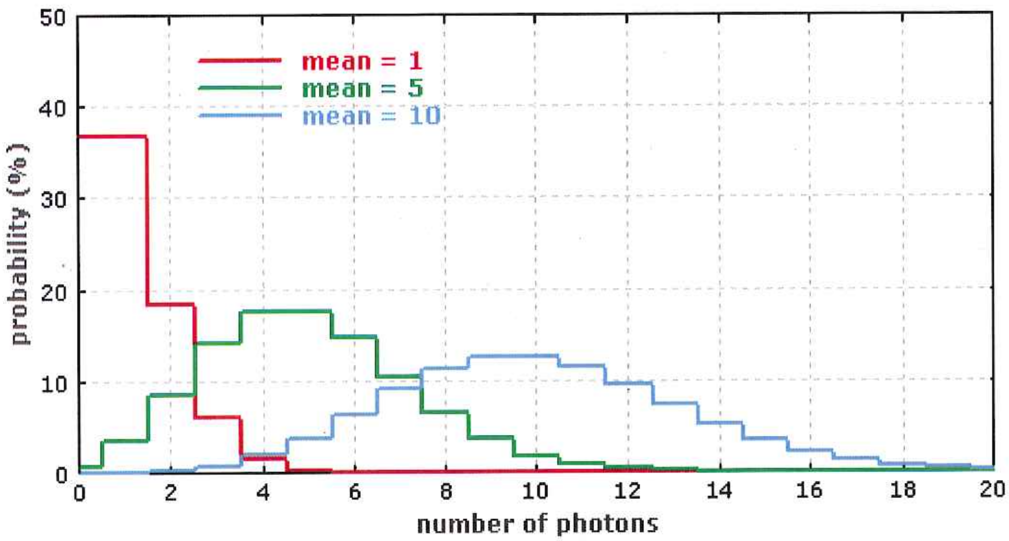
$$= \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right) (1 + 4|\alpha|^2 \cos^2(\omega t - \varphi)) \sin^2 kz$$

$$\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} = \Delta E_{vac}!$$

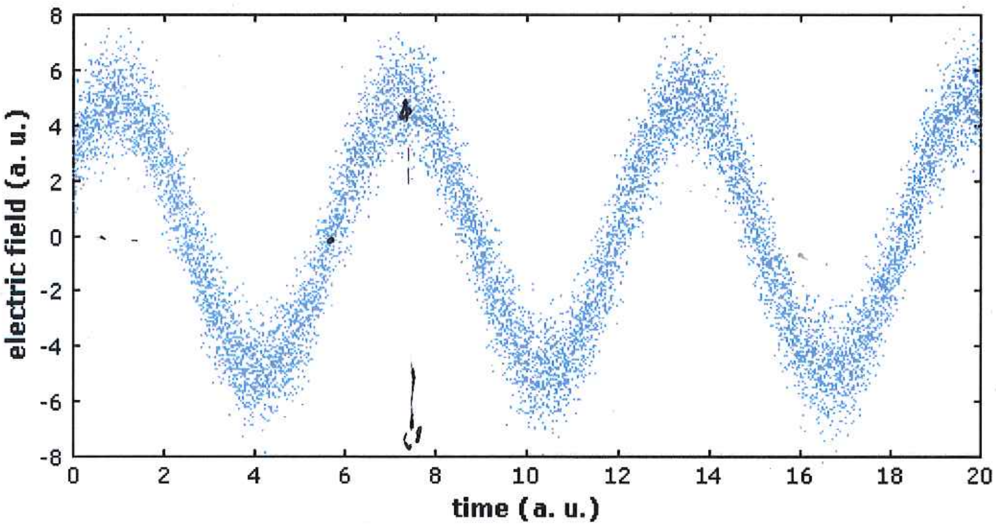
Same fluctuations as in the vacuum state!

Coherent state is a minimum-uncertainty state, and the relative uncertainty in electric field measurements decreases with mean number of photons.

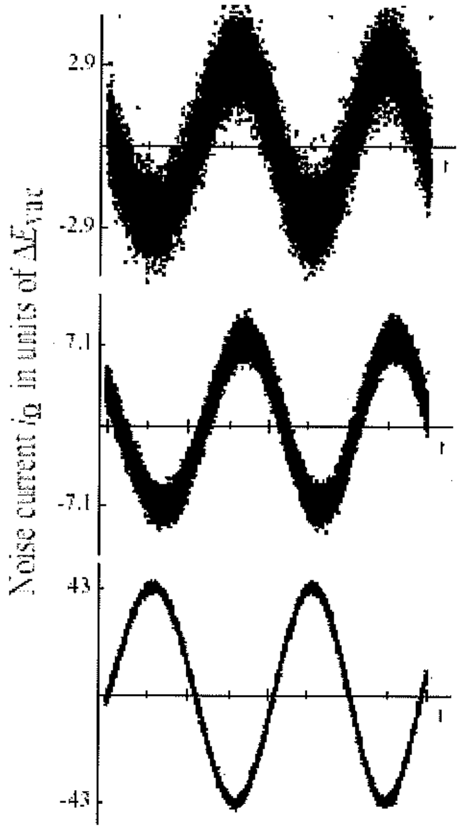
$|\alpha| \gg 1 \rightarrow$ classical E-M wave



Photon distribution in coherent states with different mean value of photons $|\alpha|^2$



Coherent state = "fuzzy" electromagnetic wave



Increasing strength of e-m field

Since the uncertainty stays the same as amplitude grows, its effect becomes less and less noticable.