A 1

$V(r) = \frac{Ze^2}{r}$ Culoumb potential

centrally-symmetric potential

$$f(\vec{k}, \vec{k}') = -\frac{2m}{\hbar^2 q} \int_0^\infty V(r') r' \sin(qr') dr' =$$

$$= -\frac{2mZe^2}{\hbar^2 q} \int_0^\infty \sin(qr') dr' = -\frac{2mZe^2}{\hbar^2 q} \lim_{\varepsilon \to 0} \mathrm{Im} \int_0^\infty e^{-\varepsilon r' + iqr'} dr' =$$

$$= -\frac{2mZe^2}{\hbar^2 q} \lim_{\varepsilon \to 0} \mathrm{Im} \frac{1}{iq - \varepsilon} = \frac{2mZe^2}{\hbar^2 q^2} = \frac{mZe^2}{2\hbar^2 k^2 \sin^2 \theta/2}$$

Differential cross-section

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{mZe^2}{\hbar^2 k^2}\right)^2 \frac{1}{(1-\cos\theta)^2}$$ Rutherford scattering

strong forward scattering $\propto \frac{1}{\theta^2}$

(notice non-zero backward scattering)

$$\sigma = \int \frac{d\sigma}{d\Omega} 2\pi \sin\theta \, d\theta \quad \propto \frac{1}{1-\cos\theta}\Big|_0^\pi = \infty$$

Since Culoumb is a long-range potential, all particles are scattered (nobody leaves unharmed)

There is no clear difference b/w slow and fast particles.

A2.

$V(r) = \frac{\alpha}{r} e^{-r/R}$ Yukawa potential

since it is centrally symmetric, we can use the following expression for the scattering amplitude

$$f(\vec{k},\vec{k}') = -\frac{2m}{\hbar^2 q}\int V(r')\cdot r' \sin qr'\, dr' =$$

$$= -\frac{2m\alpha}{q\hbar^2}\int_0^\infty e^{-r'/R}\sin qr'\, dr' = -\frac{2m\alpha}{2iq\hbar^2}\int\left[e^{-r'(1/R-iq)} - e^{-r'(\frac{1}{R}+iq)}\right]dr'$$

$$= -\frac{m\alpha}{iq\hbar^2}\left[\frac{1}{1/R-iq} - \frac{1}{1/R+iq}\right] = -\frac{2m\alpha}{\hbar^2}\frac{1}{1/R^2+q^2} = \frac{2m\alpha R^2}{\hbar^2}\frac{1}{1+q^2R^2}$$

where $q = 2k\sin\theta/2$ $\quad q^2 = 4k^2\sin^2\theta/2 = 2k^2(1-\cos\theta)$

$$\cos\theta = 1 - q^2/2k^2$$

$$d(\cos\theta) = -\frac{1}{2k^2}dq^2$$

$$\int_0^\pi \sin\theta\, d\theta \longrightarrow \frac{1}{2k^2}\int_0^{4k^2} d(q^2)$$

$$\sigma = \int_0^\pi \frac{d\sigma}{d\Omega}2\pi\sin\theta\, d\theta = \frac{2\pi}{2k^2}\left(\frac{2m\alpha R^2}{\hbar^2}\right)^2\int_0^{4k^2}\frac{d(q^2)}{(1+q^2R^2)^2} =$$

$$= \frac{\pi}{\cancel{k^2R^2}}\left(\frac{2m\alpha R^2}{\hbar^2}\right)^2\frac{4\cancel{k^2R^2}}{1+4k^2R^2} = 16\pi R^4\left(\frac{m\alpha}{\hbar^2}\right)^2\frac{1}{1+4k^2R^2}$$

Slow particles

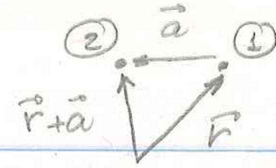
$kR \ll 1$ $\qquad \sigma = 16\pi R^4\left(\frac{m\alpha}{\hbar^2}\right)^2 \qquad \frac{d\sigma}{d\Omega} = \frac{4m^2\alpha^2R^4}{\hbar^4}$ uniform (s-wave)

Fast: $kR \gg 1$

$$\sigma = \frac{4\pi}{k^2}\left(\frac{m\alpha}{\hbar^2}\right)^2 \qquad \frac{d\sigma}{d\Omega} = \frac{4m^2\alpha^2}{\hbar^4k^4}\frac{1}{\sin^4\theta/2}$$ predominantly forward scattering

does not depend on R!

A3



$$V(\vec{r}) = V_0(r) + V_0(|\vec{r}+\vec{a}|)$$

$$\vec{q} = \vec{k} - \vec{k}'$$

$$f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\vec{r}'}\left(V_0(\vec{r}') + V_0(|\vec{r}'+\vec{a}|)\right)^2 d^3\vec{r}' =$$

$$= -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\vec{r}'} V_0(\vec{r}')\, d^3\vec{r}' - \frac{m}{2\pi\hbar^2} e^{i\vec{q}\vec{a}} \int e^{i\vec{q}(\vec{r}'+\vec{a})} V(|\vec{r}+\vec{a}|) d^3\vec{r}'$$

$$= f_0(q)\left(1 + e^{i\vec{q}\cdot\vec{a}}\right) = 2 f_0(q)\, e^{i\vec{q}\cdot\vec{a}/2} \cos \vec{q}\cdot\vec{a}/2$$

$$\frac{d\sigma}{d\Omega} = |f(\vec{k},\vec{k})|^2 = 4|f_0(q)|^2 \cos^2\left(\vec{q}\vec{a}/2\right)$$

for a particular molecular orientation $\vec{a}$

Need to average over all possible directions of $\vec{a}$:

$$\left\langle \cos^2\left(\frac{\vec{q}\cdot\vec{a}}{2}\right)\right\rangle = \frac{1}{2} + \frac{1}{2}\langle \cos \vec{q}\cdot\vec{a}\rangle$$

$$\langle \cos \vec{q}\,\vec{a}\rangle = \frac{1}{4\pi}\int \cos(qa\cos\theta')\, d\Omega' = \frac{1}{2}\int_0^{\pi} \cos(qa\cos\theta')\sin\theta\, d\theta$$

$$= \frac{1}{2qa}\sin(qax)\Big|_{-1}^{1} = \frac{\sin qa}{qa}$$

thus, averaged fractional cross-section

$$\left\langle \frac{d\sigma}{d\Omega}\right\rangle = 2|f_0(q)|^2\left(\frac{\sin qa}{qa} + 1\right) \qquad q = 2k\sin\theta/2$$

clearly, scattering pattern strongly depends on the distance b/w two atoms inside a molecule, a.

a) Slow particles $ka \ll 1 \Rightarrow qa \ll 1$

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = 4|f_0(q)|^2$$

Scattering cross-section is identical for an atom and a diatomic molecule, but σ is 4 times larger for the molecule. The actual scattering profile is determined by $|f_0(q)|^2$ - single atom scattering amplitude

b) If $qR \sim 1$, $|f_0(q)|^2$ is non-zero for $\theta \lesssim \frac{1}{2kR}$ (wide range of angles)

However, if $a \gg R$ and $kR \sim 1$, then $ka \gg 1$ and for most detection directions

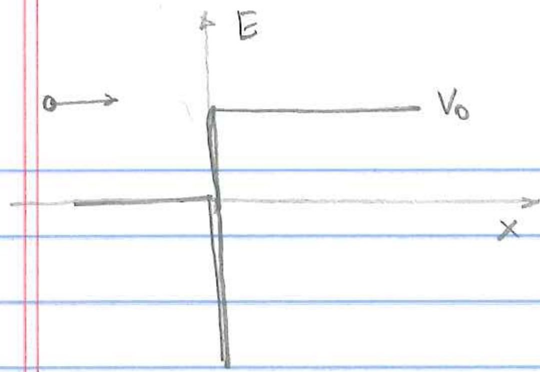
$\frac{\sin qa}{qa} \approx 0$ since $\sin qa$ is a fast-oscillating term.

Thus $$\langle \sigma \rangle_{mol} = \int \frac{d\sigma}{d\Omega} \cdot d\Omega \approx 2 \int |f_0(q)|^2 \left(1 + \cancel{\frac{\sin qa}{qa}}^{\approx 0}\right) d\Omega$$

$\approx 2 \langle \sigma \rangle_{at}$ (each atom scatters fast particle independently)

/

Q1



$$\psi(x) = \begin{cases} e^{ikx} + r e^{-ikx} & x<0 \\ t \cdot e^{ik_1 x} & x>0 \end{cases}$$

$$k = \sqrt{2mE/\hbar^2}$$
$$k_1 = \sqrt{2m(E-V_0)/\hbar^2}$$

$$\begin{cases} 1+r = t & (\psi_-(0) = \psi_+(0)) \\ ik_1 \cdot t - (1-r)(ik) = -g \cdot t & (\psi_+'(0) - \psi_-'(0) = -g\,\psi(0)) \end{cases}$$

a) $$ik_1(1+r) - (1-r)(ik) = -g(1+r)$$
$$(ik_1 + ik + g)\,r = -g + ik - ik_1$$
$$r = \frac{-g+ik-ik_1}{g+ik+ik_1} = \frac{k-k_1+ig}{k+k_1-ig}$$

Reflection coefficient $R = |r|^2 = \dfrac{(k-k_1)^2+g^2}{(k+k_1)^2+g^2}$

b) $E \gg V_0$ $\quad k_1 = \sqrt{\frac{2mE}{\hbar^2}}\sqrt{1-\frac{V_0}{E}} \simeq k\left(1-\frac{V_0}{2E}\right)$

$$\Rightarrow k-k_1 \simeq k \cdot \frac{V_0}{2E} = \sqrt{\frac{2m}{\hbar^2}}\,\frac{V_0}{\sqrt{E}} \xrightarrow{E\to\infty} 0$$

$$R(E\to\infty) = \frac{g^2}{8mE/\hbar^2} = \frac{g^2\hbar^2}{8mE} \qquad \left(\text{i.e } R \propto \frac{1}{E}\right)$$

However, if $g=0$ to begin with

$$R = \left(\frac{k-k_1}{k+k_1}\right)^2 = \left(\frac{1-k_1/k}{1+k_1/k}\right)^2 \approx \frac{1}{4}(1-k_1/k)^2 = \frac{V_0^2}{16E^2}$$

i.e $R \propto 1/E^2$

Thus, if one can measure the dependence of the reflection coefficient on E, 1/E dependence will reveal the presence of the δ-function, and $1/E^2$ will correspond to a simple step.