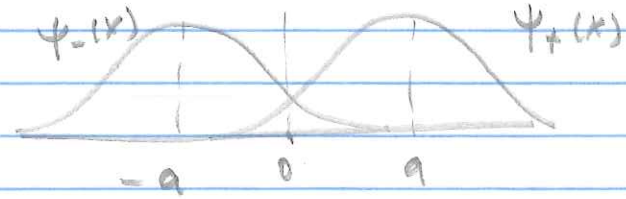
A 1.

$$\psi_\pm(x) = \sqrt[4]{\frac{\beta}{\pi}}\, e^{-\frac{\beta}{2}(x \mp a)^2}$$



Fermions — wave function is anti-symmetric : $\psi_F(x_1,x_2) = \mathcal{N}\left(\psi_+(x_1)\psi_-(x_2) - \psi_+(x_2)\psi_-(x_1)\right)$

Bosons - wave-function is symmetric:

$$\psi_B(x_1,x_2) = \mathcal{N}\left(\psi_+(x_1)\psi_-(x_2) + \psi_+(x_2)\psi_-(x_1)\right)$$

$$\mathcal{N}\iint dx_1 dx_2\, |\psi^2(x_1,x_2)| = 1$$

+ for bosons
− for fermions

$$\mathcal{N}\iint dx_1 dx_2 \left[\psi_+^2(x_1)\psi_-^2(x_2) + \psi_+^2(x_2)\psi_-^2(x_1) \pm 2\psi_+(x_1)\psi_+(x_2)\psi_-(x_1)\psi_-(x_2)\right]$$

$$= \mathcal{N}\left[2 \pm 2\left(\int_{-\infty}^{+\infty} dx_1\, \psi_+(x_1)\psi_-(x_1)\right)^2\right] = 2\mathcal{N}\left[1 \pm \left(\int_{-\infty}^{+\infty}\sqrt{\frac{\beta}{\pi}}\, e^{-\beta x^2 - \beta a^2} dx\right)^2\right]$$

$$= 2\mathcal{N}\left[1 \pm e^{-2\beta a^2}\right] = 1$$

$$\mathcal{N} = \frac{1}{\sqrt{2}\sqrt{1 \pm e^{-2\beta a^2}}}$$

$$\psi_{B,F}(x_1,x_2) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1 \pm e^{-2\beta a^2}}}\left(\psi_+(x_1)\psi_-(x_2) \pm \psi_+(x_2)\psi_-(x_1)\right)$$

$$E = \langle\psi_{B,F}|\hat{H}|\psi_{B,F}\rangle = \langle\psi_{B,F}| -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2)|\psi_{B,F}\rangle =$$

$$= -\frac{\hbar^2}{2m}\mathcal{N}^2 \iint dx_1 dx_2 \left[\psi_+(x_1)\psi_-(x_2) \pm \psi_+(x_2)\psi_-(x_1)\right]\left[\nabla^2\psi_+(x_1)\psi_-(x_2) \pm \nabla^2\psi_-(x_1)\cdot\psi_+(x_2) + (1 \leftrightarrow 2)\right]$$

$$= -\frac{\hbar^2}{2m}\mathcal{N}^2\left[\int_{-\infty}^{+\infty}\left(\psi_+(x_1)\nabla^2\psi_+(x_1) + \psi_-(x_1)\nabla^2\psi_-(x_1)\right)dx_1 \pm \int_{-\infty}^{+\infty}\psi_-(x_2)\psi_+(x_2)\,dx_2 \times\right.$$

$$\left.\times \int_{-\infty}^{+\infty}\left(\psi_+(x_1)\nabla^2\psi_-(x_1) + \psi_-(x_1)\nabla^2\psi_+(x_1)\right)dx_1\right] + \underset{\text{(identical contribution)}}{(1 \to 2)}$$

$$\nabla^2\psi_\pm = \sqrt[4]{\frac{\beta}{\pi}}\frac{d}{dx}\left(-\beta(x\mp a)e^{-\beta/2(x\mp a)^2}\right) = \sqrt[4]{\frac{\beta}{\pi}}\left(\beta^2(x\mp a)^2-\beta\right)e^{-\beta/2(x\mp a)^2}$$

$$\int_{-\infty}^{+\infty}\psi_\pm(x_1)\,\nabla^2\psi_\pm(x_1)\,dx_1 = -\beta/2$$

$$\int_{-\infty}^{+\infty}\psi_\pm(x_1)\,\nabla^2\psi_\mp(x_1)\,dx_1 = -\frac{\beta}{2}(1-2\beta a^2)\,e^{-\beta a^2}$$

$$E = -\frac{\hbar^2}{2m}N^2\left[-\beta \pm e^{-\beta a^2}\cdot(-\beta)(1-2\beta a^2)e^{-\beta a^2}\right]\times 2 =$$

$$= \frac{\hbar^2}{m}\beta N^2\left[1\pm(1-2\beta a^2)e^{-2\beta a^2}\right]$$

$$E = \frac{\hbar^2}{2m}\beta\,\frac{1\pm(1-2\beta a^2)e^{-2\beta a^2}}{1\pm e^{-2\beta a^2}}$$

(compare to $E=\frac{\hbar^2}{2m}\beta$ for distinguishable particles)

Force

$$F = -\frac{dE}{da} = -4\beta a\frac{dE}{d(2\beta a^2)} = \frac{2\hbar^2}{m}\beta^2 a\,\frac{d}{dx}\left.\frac{(1\pm(1-x)e^{-x})}{1\pm e^{-x}}\right|_{x=2\beta a^2} =$$

$$= \frac{2\hbar^2}{m}\beta^2 a\left.\left(e^{-x}\frac{e^{-x}\pm(1-x)}{(1\pm e^{-x})^2}\right)\right|_{x=2\beta a^2} =$$

$$= \frac{2\hbar^2}{m}\beta^2 a\,e^{-2\beta a^2}\,\frac{e^{-2\beta a^2}\pm(1-2\beta a^2)}{(1\pm e^{-2\beta a^2})^2}$$

For the fermions the force is always positive = repulsive

For a well-localized state $\beta a^2 \gg 1$

$$F \approx \frac{4\hbar^2}{m}\beta^3 a^3\,e^{-2\beta a^2}$$

For the fermions the force is attractive except for small separations (such that

$$e^{-2\beta a^2}+1 \geq 2\beta a^2 \Rightarrow a \leq 0.8\sqrt{\beta}$$

A2

Two particles with $\ell_{1,2} = 1$, $m_{1,2} = 0$ $\Rightarrow$
total $m = 0$, total $\ell = 0, 2$

Clebsch-Gordon coefficients

$\langle 1100 | 1100 \rangle = -1/\sqrt{3}$ $\quad\quad P_{\ell=0} = 1/3$

$\langle 1100 | 1120 \rangle = \sqrt{2/3}$ $\quad\quad P_{\ell=2} = 2/3$

$\ell = 0, 2 \rightarrow$ even parity states, spatial wavefunction is symmetric, thus the spin wavefunction must be anti-symmetric.

$$\chi = \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \quad\quad S = 0 \quad \text{singlet state}$$

$$\vec{J} = \vec{L} + \vec{S}, \quad j = 0, 2, \; m_j = 0$$

A3.

a) Identical bosons are all in the same single-particle ground state, so

$$E = NE_0$$

Only two identical fermions can occupy the same energy state, so

$$E = 2\sum_{n=0}^{N/2} E_n \quad \text{if } N \text{ is even}$$

$$E = 2\sum_{n=0}^{N-1/2} E_n + E_{N+1/2} \quad \text{if } N \text{ is odd}$$

b) Identical bosons – ψ is symmetric

" $\Psi_{123} = \psi_0(\vec{r}_1)\psi_0(\vec{r}_2)\psi_0(\vec{r}_3) \otimes \chi_{symm}$

where χ_{sym} is a symmetric spin wavefunction

$|\uparrow\uparrow\uparrow\rangle$ or $|\downarrow\downarrow\downarrow\rangle$

$$\chi_{sym} = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right)$$

$$\frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)$$

Identical fermions – two particles at $n=0$ and one at $n=1$

$$\Psi_{123} = \frac{1}{3!}\begin{vmatrix} \psi_0(\vec{r}_1)|\uparrow\rangle_1 & \psi_0(\vec{r}_1)|\downarrow\rangle_1 & \psi_1(\vec{r}_1)|\uparrow\rangle_1 \\ \psi_0(\vec{r}_2)|\uparrow\rangle_2 & \psi_0(\vec{r}_2)|\downarrow\rangle_2 & \psi_1(\vec{r}_2)|\uparrow\rangle_2 \\ \psi_0(\vec{r}_3)|\uparrow\rangle_3 & \psi_0(\vec{r}_3)|\downarrow\rangle_3 & \psi_1(\vec{r}_3)|\uparrow\rangle_3 \end{vmatrix}$$

A4

Spin-1 bosons $\longrightarrow$ wavefunction is symmetric

Spatial component $\psi(\vec{r_1})\cdot\psi(\vec{r_2})\psi(\vec{r_3})$ – symmetric

Thus, spin part must be symmetric as well

Each spin can have three possible orientations $|+\rangle \Longleftrightarrow m_s = 1$

$|-\rangle \Longleftrightarrow m_s = -1$

$|0\rangle \Longleftrightarrow m_s = 0$

1,2) $S_{total_z} = 3$: $|+++\rangle$; $S_{total_z} = -3$: $|---\rangle$

3) $S_{total_z} = 2$: $\frac{1}{\sqrt{3}}\{|++0\rangle + |+0+\rangle + |0++\rangle\}$

4) $S_{total_z} = -2$: $\frac{1}{\sqrt{3}}\{|--0\rangle + |-0-\rangle + |0--\rangle\}$

5) $S_{total_z} = 1$: $\frac{1}{\sqrt{3}}\{|+00\rangle + |0+0\rangle + |00+\rangle\}$

6) $\frac{1}{\sqrt{3}}\{|++-\rangle + |+-+\rangle + |-++\rangle\}$

7,8) $S_{total_z} = -1$ (same as 5,6, flipping "+" and "–")

9) $S_{total_z} = 0$ $|000\rangle$

10) $S_{total_z} = 0$ $\frac{1}{\sqrt{6}}\{|+-0\rangle + |+0-\rangle + |0+-\rangle + |0-+\rangle + |-+0\rangle + |-0+\rangle\}$

A5

For two particle permutation operation is equivalent to reflection with respect to their center of mass.
Thus for such system parity operator and permutation operator are identical.
Since the states with known angular momentum L have parity $(-1)^L$,
Then for $L = 0, 2, 4$ or other even number, the spin function must also be symmetric, so the total spin of the system will be even $S = 2s, 2s-2, \dots 0$,
On a contrary, if L is odd then the spin wavefunction is also odd, and
$S = 2s-1, 2s-3, \dots 1$

The situation is of coarse opposite for the fermions: even L requires odd S and vice versa