Homework #6

4.5

Weak interaction Hamiltonian

$$\hat{V} = \lambda\left[\delta^3(\vec{r})\,\vec{S}\cdot\vec{p} + \vec{S}\cdot\vec{p}\,\delta^3(\vec{r})\right]$$

From the perturbation theory

$$|n, \ell, j, m\rangle \longrightarrow |n, \ell, j, m\rangle + \sum_{n', \ell', j', m'} c_{n'\ell' j' m'} |n', \ell', j', m'\rangle$$

and

$$c_{n'\ell' j' m'} = \frac{\langle n\ell j m|\hat{V}|n'\ell' j' m'\rangle}{E - E'}$$

Since $\vec{p}$ is a vector, and $\vec{S}$ is a pseudovector, $\vec{S}\cdot\vec{p}$ is a pseudoscalar, and the $|n\ell j m\rangle$ and $|n'\ell' j' m'\rangle$ must have opposite parity : $|\ell - \ell'| = 1$

Since it is a scalar $\Delta j = 0$ $\Delta m = 0$

Also, because of the $\delta(\vec{r})$ the matrix element is not zero only if either $\psi_{n\ell m_j}(r=0) \neq 0$ or $\psi_{n'\ell' j' m'}(r=0) \neq 0$; so the weak interaction mixes only $|s\rangle$ and $|p\rangle$ states.

4.12

$\hat{H} = A\hat{S}_z^2 + B(\hat{S}_x^2 - \hat{S}_y^2)$ even under t-reversal

$[\hat{\Theta}, \hat{H}] = 0$

For the spin 1 particle

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{S}_x^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \hat{S}_y^2 = \frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \hat{S}_z^2 = \hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{H} = \hbar^2\begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

$$\hat{H}\psi = E\psi \qquad \hbar^2\begin{pmatrix} A-\lambda & 0 & B \\ 0 & -\lambda & 0 \\ B & 0 & A-\lambda \end{pmatrix} = 0$$

$$\lambda\left((A-\lambda)^2 - B^2\right) = 0$$

$\lambda_1 = 0$

$\psi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $(m_s = 0)$

$\lambda_{2,3} = A \pm B$ $\quad \psi_{2,3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|m_s = 1\rangle \pm |m_s = -1\rangle\right)$

In general, $\Theta|s, m\rangle = (-1)^m |s, -m\rangle$

$\Theta|\psi_1\rangle = \Theta|1, 0\rangle = |1, 0\rangle$ even under t-reversal

$\Theta|\psi_{2,3}\rangle = \Theta\frac{1}{\sqrt{2}}\left(|1,1\rangle \pm |1,-1\rangle\right) = (-1)|\psi_{2,3}\rangle$ odd — " —

All three states are Θ eigenstates

A1) Time-reversal symmetry

$$\hat\theta \vec S \hat\theta^{-1} = -\vec S$$

$$\hat\theta \hat{\vec p} \hat\theta^{-1} = -\vec p$$

$$\hat\theta \hat V_{weak} \theta^{-1} = -\lambda \left[\delta^3(F) \underbrace{\hat\theta \vec S \hat\theta^{-1}}_{-\vec S} \underbrace{\hat\theta \vec P \hat\theta^{-1}}_{-\vec p} + \underbrace{\hat\theta \vec S \hat\theta^{-1}}_{-\vec S} \underbrace{\hat\theta \vec P \hat\theta^{-1}}_{-\vec p} \delta(F) \right]$$

$$= \hat V_{weak}$$

$$[\hat V_{weak}, \hat\theta] = 0$$

A2

a) Initial state $|\alpha\rangle = c_1|1\rangle + c_2|2\rangle$ positive parity

$$\pi|\alpha\rangle = c_1\hat\pi|1\rangle + c_2\hat\pi|2\rangle = c_1|2\rangle + c_2|1\rangle$$

$$\parallel$$

$$|\alpha\rangle = c_1|1\rangle + c_2|2\rangle$$

Thus $c_1 = c_2 = 1/\sqrt2$

$$|\alpha\rangle = \frac{1}{\sqrt2}(|1\rangle + |2\rangle)$$

$$|\alpha(t)\rangle = \frac{1}{\sqrt2}\left(|1\rangle e^{-iE_1t/\hbar} + |2\rangle e^{-iE_2t/\hbar}\right) =$$

b) To figure out the output of the parity measurement, we need to present $|\alpha\rangle$ in terms of eigen parity states $|+\rangle$ and $|-\rangle$

$$|\pm\rangle = \frac{1}{\sqrt2}(|1\rangle \pm |2\rangle)$$

$$\langle\alpha(t)|+\rangle = \frac12\left(e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar}\right) = e^{-\frac{i(E_1+E_2)t}{\hbar}} \cos\frac{(E_1-E_2)t}{\hbar}$$

$$P_+ = \cos^2\frac{(E_1-E_2)t}{\hbar}$$

c) If $\Delta t \ll \frac{E_1-E_2}{\hbar}$, $P_+ = \cos^2\frac{(E_1-E_2)\Delta t}{\hbar} \simeq 1$, so the probability to measure negative parity $P_- \ll 1$

So in the series of measurements

$$P_+^{(N)} = (P_+)^N = \left(\cos\frac{(E_1-E_2)\Delta t}{\hbar}\right)^{2N} \approx \left[1-\left(\frac{(E_1-E_2)\Delta t}{\hbar}\right)^2\right]^N$$

Obviously for very small Δt $P_+ \approx 1$

More precisely, for $\Delta t = T/N$

$$P_+^{(N)} \approx \left(1 - \frac{1}{N^2}\left(\frac{(E_1-E_2)T}{\hbar}\right)^2\right)^N \xrightarrow{N\to\infty} \underbrace{\exp\left\{-\left(\frac{(E_1-E_2)T}{\hbar}\right)^2 \frac{1}{N}\right\}}_{\approx 1}$$

So if the state is constantly monitored, it will stay the same; in contrast, if a single measurement is done at $t = T$

$P_+ = \cos^2 \frac{(E_1-E_2)T}{\hbar}$, and the system can deterministically flip into the negative parity state for $T = \frac{\pi\hbar}{2(E_1-E_2)}$

A3) According to Wigner-Eckard theorem, the expectation value of $\langle \vec{d}_a \rangle$ must be proportional to the total angular momentum $\langle \vec{J} \rangle$, thus $\vec{d}_a$ is transformed as pseudo-vector under the time-reversal. At the same time, $\vec{E}$ is a (normal) vector

$$\hat{\theta} \vec{E} \hat{\theta}^{-1} = \vec{E} \qquad \hat{\theta} \vec{d}_a \hat{\theta}^{-1} = -\vec{d}_a$$

$\hat{H} = -\vec{d}_a \cdot \vec{\mathcal{E}}$ is odd under time reversal

Similarly $\hat{\pi} \vec{d}_a \hat{\pi}^{-1} \propto \hat{\pi} \vec{J} \hat{\pi}^{-1} = \vec{J}$ even

$\hat{\pi} \vec{E} \hat{\pi}^{-1} = -\vec{E}$ odd

Thus $\hat{H}$ is odd under parity transformation

Q1

$$\hat{V} = V_0 |x|$$

$$\psi(x) = \sqrt{\alpha}\, e^{-\alpha|x|}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0|x|$$

$$\bar{H} = -\frac{\hbar^2\alpha}{2m}\int_{-\infty}^{+\infty} e^{-\alpha|x|}\frac{d^2}{dx^2}\left(e^{-\alpha|x|}\right)dx + \alpha V_0 \int_{-\infty}^{+\infty} |x|\, e^{-2\alpha|x|}\,dx =$$

$$= -\frac{\hbar^2\alpha^2}{2m}\int_{-\infty}^{+\infty} e^{-|t|}\frac{d^2}{dt^2}\left(e^{-|t|}\right)dt + \frac{V_0}{\alpha}\int_{-\infty}^{+\infty} |t|\, e^{-2|t|}\,dt$$

$$\frac{d}{dt} e^{-|t|} = -(\theta(t) - \theta(-t))\, e^{-|t|}$$

$$\frac{d^2}{dt^2} e^{-|t|} = -2\delta(t)\, e^{-|t|} + e^{-|t|}$$

$$\int_{-\infty}^{+\infty}\left[(-2\delta(t)+1\right] e^{-2|t|}\,dt = -1 \qquad \int_{-\infty}^{+\infty}|t|\, e^{-2|t|}\,dt = \frac{1}{2}\Gamma(2) = \frac{1}{2}$$

$$\bar{H} = \frac{\hbar^2\alpha^2}{2m} + \frac{V_0}{2\alpha} \qquad \frac{\partial\bar{H}}{\partial\alpha} = \frac{\hbar^2\alpha}{m} - \frac{V_0}{2\alpha^2} = 0$$

$$\alpha_{min} = \sqrt[3]{\frac{mV_0}{2\hbar^2}} \qquad E_0 \simeq \frac{3}{2^{5/3}}\sqrt[3]{\frac{V_0^2\hbar^2}{m}}$$