Homework #5

5.23

$$F = F_0 e^{-t/\tau}$$
$$V = -xF = -xF_0 e^{-t/\tau}$$

a) $V_{10} = -x_{10} \cdot F_0 e^{-t/\tau} = \sqrt{\frac{\hbar}{2m\omega}} F_0 e^{-t/\tau}$

$$c_1 = -\frac{i}{\hbar} \int_0^t V_{10}(t) e^{i\omega t'} dt' = \sqrt{\frac{\hbar}{2m\omega}} F_0 \frac{i}{\hbar} \int_0^t e^{(i\omega - 1/\tau)t'} dt' =$$

$$= i \sqrt{\frac{1}{2m\omega\hbar}} F_0 \frac{e^{(i\omega - 1/\tau)t} - 1}{i\omega - 1/\tau}$$

$$P_1 = \frac{F_0^2 \tau^2}{2m\omega\hbar} \frac{1 + e^{-2t/\tau} - 2\cos\omega t \, e^{-t/\tau}}{1 + \omega^2\tau^2} \longrightarrow \frac{F_0^2\tau^2}{2m\omega\hbar} \frac{1}{1+\omega^2\tau^2}$$

since the perturbation acts only in the timescale $t \lesssim \tau$, we expect the system to stabilize in some new state at longer time, when it becomes negligeable.

b) In the first order no states with $n \geq 2$ can be excited since $V_{n0} = 0$.
However, the second-order perturbation will produce non-zero P_2, and in general n-th order perturbation must be used to calculate P_n.

5.32

$$H = \underbrace{A\, S_1 \cdot S_2}_{\hat{H}_0} + \underbrace{\left(\frac{eB}{m_e c}\right)(S_{1z} - S_{2z})}_{V}$$

Unperturbed hamiltonian: $\hat{H}_0 = A\,\vec{S}_1 \cdot \vec{S}_2 = \frac{A}{2}\left[(\vec{S}_1+\vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2\right]$

$\hat{H}_0 = A\left(\hat{S}^2 - 3/2\,\hbar^2\right)$ where $\vec{S} = \vec{S}_1 + \vec{S}_2$

$S = 1$ $\quad E_{1,2,3} = \frac{A\hbar^2}{4}$ $\quad m_S = \begin{cases} +1 & |\uparrow\uparrow\rangle \quad |3\rangle \\ 0 & \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \quad |2\rangle \\ -1 & |\downarrow\downarrow\rangle \quad |1\rangle \end{cases}$

$S = 0$ $\quad E_0 = -\frac{3A\hbar^2}{4}$ $\quad m_S = 0 \quad \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad |0\rangle$

In this basis non-zero matrix elements are $V_{20} = V_{02} = \frac{eB\hbar}{m_e c}$ and all diagonal elements are zero

Second order perturbation:

$\Delta E_1 = \Delta E_3 = 0$

$$\Delta E_2 = \frac{|V_{20}|^2}{A\hbar^2} = \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2 \Rightarrow E_2 = \frac{A\hbar^2}{4} + \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2$$

$$\Delta E_0 = -\frac{|V_{0}|^2}{A\hbar^2} = -\frac{1}{A}\left(\frac{eB}{m_e c}\right)^2 \Rightarrow E_0 = -\frac{3A\hbar^2}{4} - \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2$$

Since such treatment is valid for $\langle V\rangle \ll \langle H_0\rangle$, so $\hbar eB/m_e c \ll A\hbar^2$, then the exact solution in this limit is

$$E_{0,2} \approx -\left(\frac{A\hbar^2}{4}\right)\left[1 \pm 2\left(1 + 2\left(\frac{eB}{m_e c\,\hbar A}\right)^2\right)\right]$$

$$E_0 \approx -\frac{3A\hbar^2}{4} - \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2$$

$$E_2 \approx \frac{A\hbar^2}{4} + \frac{1}{A}\left(\frac{eB}{m_e c}\right)^2$$

coinside with the perturbation theory prediction

b) In order to induce transitions between two states, we must have non-zero matrix element, $\langle 0 | \hat{V}_{int} | 2 \rangle$

$$\hat{V}_{int} = - \vec{\mu} \vec{B} \cos\omega t = - \frac{e}{m_e c} \vec{B} (\vec{S}_1 - \vec{S}_2)$$

if $\vec{B} = \vec{e}_z B_0$, then $V_{int} = - \frac{eB_0}{m_e c} \cos\omega t (S_{1z} - S_{2z})$

if $\vec{B} = \vec{e}_{x,y} B_0$, then $V_{int} = - \frac{eB_0}{m_e c} \cos\omega t (S_{1x,y} - S_{2x,y})$

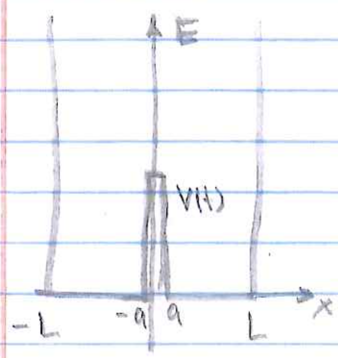
Since for transitions b/w $|0\rangle$ and $|2\rangle$ $\Delta M = 0$, then only S_z will have non-zero matrix elements ($S_{x,y} \propto S_+ \pm S_-$)
Thus, the oscillating field must be directed in z-direction.

c) $$|0\rangle^{(1)} = |0\rangle^{(0)} + \frac{V_{20}}{E_0 - E_2} |2\rangle^{(0)} = |0\rangle^{(0)} - \frac{eB}{m_e c A \hbar} |2\rangle^{(0)}$$

$$|2\rangle^{(1)} = |2\rangle^{(0)} + \frac{V_{02}}{E_2 - E_0} |0\rangle^{(0)} = |2\rangle^{(0)} + \frac{eB}{m_e c A \hbar} |0\rangle^{(0)}$$

~~Problem 1~~

A1



Unperturbed states

$$E_n^{(0)} = \frac{\pi^2 \hbar^2 n^2}{8mL^2}$$

$$\psi_n^{(0)} = \sqrt{\frac{1}{L}} \begin{cases} \cos \frac{\pi n x}{2L} & n \text{ is odd} \\ \sin \frac{\pi n x}{2L} & n \text{ is even} \end{cases}$$

Transition matrix elements

$$V_{1n} = \langle 1|V|n\rangle = \frac{V_0}{\sqrt{L}} \int_{-a}^{a} \cos \frac{\pi x}{2L} \psi_n^{(0)}(x)\, dx$$

if n is odd

$$V_{1n} = \frac{V_0}{L} \int_{-a}^{a} \cos \frac{\pi n x}{2L} \cos \frac{\pi x}{2L}\, dx \approx \frac{2aV_0}{L} \qquad \left(\frac{a}{L} \ll 1\right)$$

if n is even

$$V_{1n} = \frac{V_0}{L} \int_{-a}^{a} \sin \frac{\pi n x}{2L} \cos \frac{\pi x}{2L}\, dx \approx \frac{\pi n V_0}{2L^2} \int_{-a}^{a} x\, dx = \frac{\pi n V_0 a^2}{4L^2}$$

Then the probability of finding a particle in the n-th state is

$$P_n(t) = \left| -\frac{i}{\hbar} \int_0^t V_{1n} e^{-t'/\tau} e^{i\omega_{1n} t'} dt' \right|^2 =$$

$$= \frac{|V_{1n}|^2}{\hbar^2} \frac{\left|e^{-t/\tau + i\omega_{1n} t} - 1\right|^2}{1/\tau^2 + \omega_{1n}^2}$$

For $t \ll \tau$ $\quad e^{-t/\tau} \approx 1$ $\quad P_n(t) \simeq \dfrac{|V_{1n}|^2 \tau^2}{\hbar^2 (1+\omega_{1n}^2 \tau^2)} \sin^2 \dfrac{\omega_{1n} t}{2}$

For $t \gg \tau$ $\quad e^{-t/\tau} \to 0$ $\quad P_n(t) \simeq \dfrac{|V_{1n}|^2 \tau^2}{\hbar^2 (1+\omega_{1n}^2 \tau^2)}$

Transidion to all even-numbered states are suppressed by the factor $(^a/_L)^2$ (parity)

For short times the probabilities are time-dependent, $P_n(t) \sim t^2$ for very short times, then oscillates with $\omega_{1n}/2$. After long time (effectively after the perturbation is off) the state distribution is constant, with $P_n \sim 1/n^4$ dependence for odd states, and $P_n \sim 1/n^2$ for even states.

A2 ~~Problem 2~~

$$\hat{V}_{int} = -\vec{\mu}\vec{B} - \vec{d}\vec{E} = \overbrace{\tfrac{\mu_B}{\hbar} L_z B_0}^{\hat{V}_B} \overbrace{- e z E_0}^{\hat{V}_E}$$

Magnetic field shifts only the states with $m_\ell = \pm 1$ $(\ell = 1)$

$\langle n \ell m | V_B | n \ell m \rangle = \mu_B m B_0$

Electric field requires $\Delta \ell = 1$ $\Delta m = 0$, so it mixes $|210\rangle$ and $|200\rangle$ states

$$\langle 210 | z | 200 \rangle = \frac{2\pi}{32\pi a_0^3} \int_0^\infty \left(2 - \frac{r}{a_0}\right) \frac{r^2}{a_0} e^{-r/a_0} r^2 dr \int_0^\pi \cos^2\theta \, d\cos\theta$$

$$= \frac{a_0}{16} \frac{2}{3} \int_0^\infty (2-x) x^4 e^{-x} dx = \frac{a_0}{24} (2 \cdot 4! - 5!) = -3a_0$$

$\langle 210 | -ezE_0 | 200 \rangle = 3 e a_0 E_0$

Degenerate two-level perturbation theory

$$\hat{V} = \begin{pmatrix} 0 & 3ea_0E_0 \\ 3ea_0E_0 & 0 \end{pmatrix} \begin{array}{l} |200\rangle \\ |210\rangle \end{array}$$

$$\begin{vmatrix} -\lambda & 3ea_0E_0 \\ 3ea_0E_0 & -\lambda \end{vmatrix} = 0 \qquad \lambda_{1,2} = \pm 3ea_0E_0$$

$$|1,2\rangle = \frac{1}{\sqrt{2}}\left(|200\rangle \pm |210\rangle\right)$$

So in general

$|211\rangle$

$n=2$ $\quad |2\rangle = \frac{1}{\sqrt{2}}\left(|200\rangle - |210\rangle\right)$

$|1\rangle = \frac{1}{\sqrt{2}}\left(|200\rangle + |210\rangle\right)$

$|21-1\rangle$

Degeneracies remain in a) $E_0 = 0$ or b) $B_0 = 0$ or c) $\mu_B B_0 = \pm 3 e a_0 E_0$

~~Problem 4~~

A3 In regions with $\vec{B}_z$ field only

$$\hat{H}_0 = \left(\frac{\hat{p}^2}{2m}\right) - \mu_n B_0 \hat{\sigma}_z$$

at $t=0, y=0$ $|\alpha_0\rangle = |{}^1_0\rangle$

In the region with $\vec{B}_z$ and $\vec{B}_x$ fields

$$\hat{H} = \left(\frac{\hat{p}^2}{2m}\right) - \mu_n B_0 \hat{\sigma}_z - \mu_n B_1 \hat{\sigma}_x$$

New eigen-functons

$$\hat{H}|\alpha\rangle = \lambda|\alpha\rangle \qquad |\alpha\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$$

$$-\mu_n B_0 \left\{a\begin{pmatrix}1\\0\end{pmatrix} - b\begin{pmatrix}0\\1\end{pmatrix}\right\} - \mu_n B_1 \left\{a\begin{pmatrix}0\\1\end{pmatrix} + b\begin{pmatrix}1\\0\end{pmatrix}\right\} = \lambda\left(a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$-\mu_n B_0 a - \mu_n B_1 b = \lambda a \qquad b = \frac{-\mu_n B_0 - \lambda}{\mu_n B_1} a$$

$$\mu_n B_0 b - \mu_n B_1 a = \lambda b \qquad a = \frac{\mu_n B_0 - \lambda}{\mu_n B_1} b$$

$$(\mu_n B_0)^2 - \lambda^2 = (\mu_n B_1)^2 \Rightarrow \lambda = \pm \mu_n \sqrt{B_0^2 + B_1^2} \equiv \pm \mu_n B'$$

(could have guessed!)

$$|\alpha_{1,2}\rangle = N_{1,2}\left[\begin{pmatrix}1\\0\end{pmatrix} - \frac{B_0 \pm B'}{B_1}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$N_{1,2}$ – normalization constants

At $t=0$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 N_1 \begin{pmatrix} 1 \\ -\frac{B_0+B'}{B_1} \end{pmatrix} + c_2 N_2 \begin{pmatrix} 1 \\ -\frac{B_0-B'}{B_1} \end{pmatrix}$$

$$c_1 N_1 + c_2 N_2 = 1$$

$$(c_1 N_1 + c_2 N_2)\left(-\frac{B_0}{B_1}\right) + (c_2 N_2 - c_1 N_1)\left(\frac{B'}{B_1}\right) = 0$$

$$c_2 N_2 - c_1 N_1 = \frac{B_0}{B'}$$

$$c_1 N_1 = \frac{1}{2}\left(1 - \frac{B_0}{B'}\right) \quad c_2 N_2 = \frac{1}{2}\left(1 + \frac{B_0}{B'}\right)$$

Thus, the state of the system at time t

$$|\alpha(t)\rangle = \frac{1}{2}\left(1 - \frac{B_0}{B'}\right)\begin{pmatrix} 1 \\ -\frac{B_0+B'}{B_1} \end{pmatrix} e^{-\frac{i\mu_n B' t}{\hbar}} + \frac{1}{2}\left(1 + \frac{B_0}{B'}\right)\begin{pmatrix} 1 \\ -\frac{B_0-B'}{B_1} \end{pmatrix} e^{\frac{i\mu_n B' t}{\hbar}}$$

$$= \begin{pmatrix} \cos\frac{\mu_n B' t}{\hbar} + i\frac{B_0}{B'}\sin\frac{\mu_n B' t}{\hbar} \\ i\frac{B_1}{B'}\sin\frac{\mu_n B' t}{\hbar} \end{pmatrix}$$

The probability of the spin flip at $t = t_0$

$$P_\downarrow(t_0) = \frac{B_1^2}{B_0^2 + B_1^2}\sin^2\left(\frac{\mu_n\sqrt{B_1^2+B_0^2}\,t}{\hbar}\right)$$