Spring 2016

Homework #1 (Solutions)

Q1. The problem is about time evolution of energy eigenstates

$$\hat{H} = \hat{H}_0 - \omega \hat{L}_z$$

Eigenstates with $n=2$, $l=1$ ($\hat{H}_0 \psi_{21} = E_{21}\psi_{21}$)

$m=0$ $\quad E_0 = E_{21}$

$m=\pm 1$ $\quad E_\pm = E_{21} \mp \hbar\omega$

At $t=0$ $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\left(|21-1\rangle - |211\rangle\right)$

Time evolution $\quad |\psi(t)\rangle = |\psi(0)\rangle e^{-iEt/\hbar}$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|21-1\rangle e^{-i(E_{21}+\hbar\omega)t/\hbar} - |211\rangle e^{-\frac{i(E_{21}-\hbar\omega)t}{\hbar}}\right)$$

$$= \frac{1}{\sqrt{2}} e^{-iE_{21}t/\hbar}\left(|21-1\rangle e^{-i\omega t} - |211\rangle e^{i\omega t}\right)$$

Probabilities of each particular state

$$P_{px} = |\langle 2p_x|\psi(t)\rangle|^2 = \left|\frac{1}{2}\left(\langle 21-1| - \langle 211|\right)\left(|21-1\rangle e^{-i\omega t} - |211\rangle e^{i\omega t}\right)\right|^2$$

$$= \left|\frac{1}{2}\left(e^{-i\omega t} + e^{i\omega t}\right)\right|^2 = \cos^2\omega t$$

$$P_{py} = |\langle 2p_y|\psi(t)\rangle|^2 = \left|\frac{1}{2}\left(\langle 21-1| + \langle 211|\right)\left(|21-1\rangle e^{-i\omega t} - |211\rangle e^{i\omega t}\right)\right|^2 =$$

$$= \left|\frac{1}{2}\left(e^{-i\omega t} - e^{i\omega t}\right)\right|^2 = \sin^2\omega t$$

$P_{pz} = 0$

$P_{px} = 1$ at $t = 0, \frac{\pi}{\omega}, \frac{2\pi}{\omega} \ldots$

$P_{py} = 1$ at $t = \frac{\pi}{2\omega}, \frac{3\pi}{\omega}, \ldots$

P_{py} is always zero

Q2

$$|\psi\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}$$

$$\vec{n} = \sin\theta'\cos\varphi'\,\vec{e}_x + \sin\theta'\sin\varphi'\,\vec{e}_y + \cos\theta'\,\vec{e}_z$$

$$\vec{n}\vec{S} = \frac{\hbar}{2}\left(\sin\theta'\cos\varphi'\,\hat{\sigma}_x + \sin\theta'\sin\varphi'\,\hat{\sigma}_y + \cos\theta'\,\hat{\sigma}_z\right) =$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta' & \sin\theta' e^{-i\varphi'} \\ \sin\theta' e^{i\varphi} & -\cos\theta' \end{pmatrix}$$

$$\vec{n}\vec{S}|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} \cos\theta' & \sin\theta' e^{-i\varphi'} \\ \sin\theta' e^{i\varphi'} & -\cos\theta' \end{pmatrix}\begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}$$

Thus

$$\cos\theta'\cos\theta + \sin\theta'\sin\theta\, e^{i(\varphi-\varphi')} = \cos\theta$$

$$\sin\theta'\cos\theta e^{i\varphi'} - \cos\theta'\sin\theta\, e^{i\varphi} = e^{i\varphi}\sin\theta$$

From the top equation it is clear that $e^{i(\varphi-\varphi')}$ must be real (i.e $= 1$ or -1)

a) For $\varphi = \varphi'$

$$\cos\theta'\cos\theta + \sin\theta'\sin\theta = \cos(\theta-\theta') = \cos\theta \Rightarrow \theta' = 2\theta$$

b) For $\varphi - \varphi' = \pi$

$$\cos\theta'\cos\theta - \sin\theta'\sin\theta = \cos(\theta+\theta') = \cos\theta \Rightarrow \theta' = -2\theta$$

(Which is an opposite direction from a))

For $\hat{S}_x$ $\vec{n} = (1,0,0)$ $\theta' = \pi/2$ $\varphi' = 0$ $\Rightarrow \theta = \frac{\theta'}{2} = \frac{\pi}{4}$

$$|\psi_{x+}\rangle = \begin{pmatrix} \cos\pi/4 \\ \sin\pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\hat{S}_y$ $\vec{n} = (0,1,0)$ $\theta' = \pi/2$ $\varphi' = \pi/2$ $\Rightarrow \theta = \pi/4,\ \varphi = \pi/2$

$$|\psi_{y+}\rangle = \begin{pmatrix} \cos\pi/4 \\ e^{i\pi/2}\sin\pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

For the negative eignen value $\theta' \to \theta' + \pi$, $\varphi' \to \varphi'$

$$|\psi_{x-}\rangle = \begin{pmatrix} \cos 3\pi/4 \\ \sin 3\pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix};\ |\psi_{y-}\rangle = \begin{pmatrix} \cos 3\pi/4 \\ e^{i\pi/2}\sin 3\pi/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix}$$

Q3. $|\langle lm|\hat{H}_M + \hat{H}_E|l'm'\rangle|^2 = (\langle lm|\hat{H}_M|l'm'\rangle + \langle lm|\hat{H}_E|l'm'\rangle)\times$

$\times [(\langle lm|\hat{H}_M|l'm'\rangle)^* + \langle lm|\hat{H}_E|l'm'\rangle)^* =$

$= |\langle lm|\hat{H}_M|l'm'\rangle|^2 + |\langle lm|\hat{H}_E|l'm'\rangle|^2 + \langle lm|\hat{H}_M|l'm'\rangle\langle lm|\hat{H}_E|l'm'\rangle^*$

$+ \langle lm|\hat{H}_M|l'm'\rangle^* \langle lm|\hat{H}_E|l'm'\rangle$

From the form of the spherical function

$Y^*_{lm} = (-1)^m Y_{l-m} \Rightarrow |lm\rangle^* = (-1)^m |l-m\rangle$

$(\langle lm|H_M|l'm'\rangle)^* = (-1)^{m+m'} \langle l-m|H^*_M|l'-m'\rangle =$

$= (-1)^{m+m'}\left(-\frac{q}{2\mu c}\vec{B}\right)\langle l-m|\underbrace{\hat{\vec{L}}^*}_{=-\hat{\vec{L}}}|l'-m'\rangle = (-1)^{m+m'}\left(\frac{q}{2\mu c}\vec{B}\right)\times$

$\times \langle l-m|\hat{\vec{L}}|l'-m'\rangle$

Similarly

$\langle lm|\hat{H}_E|l'm'\rangle^* = (-1)^{m+m'}\langle l-m|H^*_E|l'-m'\rangle =$

$= (-1)^{m+m'}(-q\vec{E})\langle l-m|\vec{r}|l'-m'\rangle$

The remaining step is to connect $\langle l-m|..|l'-m'\rangle$ and $\langle lm|..|l'm'\rangle$ matrix elements.

Clearly, we can do that by rotating the coordinate system such that $z\to -z$. It is convenient to choose the x axis such that it is perpendicular to both $\vec{\mathcal{E}}$ and $\vec{B}$, and rotate the system around it.

Then $x\to x$, $y\to -y$, $z\to -z$

and $|lm\rangle \to |l-m\rangle$

Clearly both $\vec{B}\cdot\vec{L}$ and $\vec{\mathcal{E}}\cdot\vec{r}$ both change sign under such rotation, and

$\langle l-m|\vec{B}\vec{L}|l'-m'\rangle = -\langle lm|\vec{B}\vec{L}|l'm'\rangle$

$\langle l-m|\vec{\mathcal{E}}\vec{r}|l'-m'\rangle = -\langle lm|\vec{\mathcal{E}}\vec{r}|l'm'\rangle$

The bottom line is

$$\langle lm|H_M|l'm'\rangle^* = (-1)^{m+m'} \langle lm|H_m|l'm'\rangle$$

and

$$\langle lm|H_E|l'm'\rangle^* = -(-1)^{m+m'} \langle lm|H_E|l'm'\rangle$$

And thus

$$\langle lm|\hat{H}_M|l'm'\rangle\langle lm|\hat{H}_E|l'm'\rangle^* + \langle lm|\hat{H}_M|l'm'\rangle^*\langle lm|\hat{H}_E|l'm'\rangle = 0$$

Also since $\hat{\vec{L}} = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)\vec{e}_x + \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)\vec{e}_y + \hat{L}_z \vec{e}_z$

$\langle lm|\hat{\vec{B}}\hat{\vec{L}}|l'm'\rangle = 0$ for any $l \neq l'$

At the same time $\vec{r} = r\cdot\sqrt{\frac{4\pi}{3}}\left(Y_{10}\vec{e}_z + \frac{Y_{1-1}-Y_{11}}{\sqrt{2}}\vec{e}_x + \vec{e}_z\frac{iY_{11}+iY_{1-1}}{\sqrt{2}}\right)$

and $\langle lm|\vec{\mathcal{E}}\vec{r}|l'm'\rangle = 0$ for any $|l-l'| \neq 1$

Later we will discuss how we can make this argument from the point of view of the operators' parity.

Q4

$$\hat{H} = -\frac{e^2 g_1 g_2}{2m_1 m_2}\alpha\left(\vec{S}^{(1)}\vec{S}^{(2)}\right) - \frac{e}{2m_1 m_2}\vec{B}\left(m_2 g_1 \vec{S}_1^{(1)} - m_1 g_2 \vec{S}^{(2)}\right)$$

Since $\vec{B}$ is the only fixed direction, it is convenient to direct z-axis along $\vec{B}$

Also, it is convenient to use $\hat{S}_\pm$ operators

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)} = \frac{1}{2}\left(\hat{S}_+^{(1)}\hat{S}_-^{(2)} + \hat{S}_-^{(1)}\hat{S}_+^{(2)}\right) + \hat{S}_z^{(1)}\hat{S}_z^{(2)}$$

Then

$$\hat{H} = -a\left(\hat{S}_+^{(1)}\hat{S}_-^{(2)} + \hat{S}_-^{(1)}\hat{S}_+^{(2)}\right) - 2a\hat{S}_z^{(1)}\hat{S}_z^{(2)} - 2b_1\hat{S}_z^{(1)} + 2b_2\hat{S}_z^{(2)}$$

$$\hat{S}_+|\uparrow\rangle = 0 \qquad \hat{S}_+|\downarrow\rangle = \hbar|\uparrow\rangle$$

$$\hat{S}_-|\uparrow\rangle = \hbar|\downarrow\rangle \qquad \hat{S}_-|\downarrow\rangle = 0$$

$$\hat{H}|\uparrow_1\uparrow_2\rangle = -\frac{a\hbar^2}{2}|\uparrow_1\uparrow_2\rangle - \hbar b_1|\uparrow_1\uparrow_2\rangle + \hbar b_2|\uparrow_1\uparrow_2\rangle \quad \text{eigenstate}$$

$$\hat{H}|\downarrow_1\downarrow_2\rangle = -\frac{a\hbar^2}{2}|\downarrow_1\downarrow_2\rangle + \hbar b_1|\downarrow_1\downarrow_2\rangle - \hbar b_2|\downarrow_1\downarrow_2\rangle \quad \text{eigenstate}$$

$$\left.\begin{aligned}\hat{H}|\uparrow_1\downarrow_2\rangle &= -a\hbar^2|\downarrow_1\uparrow_2\rangle + \left(\frac{a\hbar^2}{2} - \hbar b_1 - \hbar b_2\right)|\uparrow_1\downarrow_2\rangle \\ \hat{H}|\downarrow_1\uparrow_2\rangle &= -a\hbar^2|\uparrow_1\downarrow_2\rangle + \left(\frac{a\hbar^2}{2} + \hbar b_1 + \hbar b_2\right)|\downarrow_1\uparrow_2\rangle\end{aligned}\right\}\text{mixed}$$

$$\left[\begin{aligned}E_{\uparrow\uparrow} &= -\frac{a\hbar^2}{2} - \hbar b_1 + \hbar b_2 & |\psi_{\uparrow\uparrow}\rangle &= |\uparrow_1\uparrow_2\rangle \\ E_{\downarrow\downarrow} &= -\frac{a\hbar^2}{2} + \hbar b_1 - \hbar b_2 & |\psi_{\downarrow\downarrow}\rangle &= |\downarrow_1\downarrow_2\rangle\end{aligned}\right.$$

Clearly, the other two states are the combination of $|\uparrow_1\downarrow_2\rangle$ and $|\downarrow_1\uparrow_2\rangle$ $\quad |\psi\rangle = A|\uparrow_1\downarrow_2\rangle + B|\downarrow_1\uparrow_2\rangle$

$$\hat{H}|\psi\rangle = -a\hbar^2\left(A|\downarrow_1\uparrow_2\rangle + B|\uparrow_1\downarrow_2\rangle\right) + \frac{a\hbar^2}{2}\left(A|\uparrow_1\downarrow_2\rangle + B|\downarrow_1\uparrow_2\rangle\right) -$$

$$-(\hbar b_1 + \hbar b_2)\left(A|\uparrow_1\downarrow_2\rangle - B|\downarrow_1\uparrow_2\rangle\right) = E\left(A|\uparrow_1\downarrow_2\rangle + B|\downarrow_1\uparrow_2\rangle\right)$$

$$\begin{cases} -a\hbar^2 B + \left(\frac{a\hbar^2}{2} - \hbar b_1 - \hbar b_2\right) A = E \cdot A \\ -a\hbar^2 A + \left(\frac{a\hbar^2}{2} + \hbar b_1 + \hbar b_2\right) B = E B \end{cases}$$

$$-\frac{a\hbar^2 B}{A} + \cancel{\frac{a\hbar^2}{2}} - \hbar b_1 - \hbar b_2 = -\frac{a\hbar^2 A}{B} + \cancel{\frac{a\hbar^2}{2}} + \hbar b_1 + \hbar b_2$$

Since $A^2 + B^2 = 1$ $\quad A = \sin\gamma, \; B = \cos\gamma, \; \frac{A}{B} = \tan\gamma = x$

$$-a\hbar^2 \frac{1}{x} + a\hbar^2 \cdot x - 2\hbar(b_1 + b_2) = 0$$

$$x^2 - 2\,\frac{b_1 + b_2}{a\hbar}\,x - 1 = 0$$

$$x_{1,2} = \frac{b_1 + b_2}{a\hbar} \pm \sqrt{1 + \left(\frac{b_1 + b_2}{a\hbar}\right)^2} = \tan\gamma_\pm$$

$$E_\pm = \frac{a\hbar^2}{2} + \cancel{\hbar(b_1 + b_2)} - \cancel{\hbar(b_1 + b_2)} \mp \hbar\sqrt{(a\hbar)^2 + (b_1 + b_2)^2} =$$

$$\left[\begin{array}{l} E_\pm = \frac{a\hbar^2}{2}\left(1 \mp \sqrt{1 + \left(\frac{b_1 + b_2}{a\hbar}\right)^2}\right) \\ |\psi_\pm\rangle = \sin\gamma_\pm \, |\uparrow_1 \downarrow_2\rangle + \cos\gamma_\pm \, |\downarrow_1 \uparrow_2\rangle \end{array}\right.$$

Q5

$$\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}\left(J^2 - J_1^2 - J_2^2\right)$$

$$\vec{J}_1 \cdot \vec{J}_2 \;|j_1, j_2; j\, m\rangle = \frac{\hbar^2}{2}\left(j(j+1) - j_1(j_1+1) - j_2(j_2+1)\right)|j_1 j_2; j\, m\rangle$$

$$\vec{J}_1 \cdot \vec{J} = \vec{J}_1 \cdot (\vec{J}_1 + \vec{J}_2) = J_1^2 + \vec{J}_1 \vec{J}_2 = \frac{1}{2}\left(J^2 + J_1^2 - J_2^2\right)$$

$$\vec{J}_1 \cdot \vec{J} \;|j_1 j_2; j\, m\rangle = \frac{\hbar^2}{2}\left(j(j+1) + j_1(j_1+1) - j_2(j_2+1)\right)|j_1 j_2; j\, m\rangle$$

Similarly

$$\vec{J}_2 \cdot \vec{J} \;|j_1 j_2; j\, m\rangle = \frac{\hbar^2}{2}\left(j(j+1) + j_2(j_2+1) - j_1(j_1+1)\right)|j_1, j_2; j\, m\rangle$$