Homework 10

A1 $V(r) = -\frac{\hbar^2 \lambda}{2\mu}\delta(r-R)$

Solution of the radial part of the wavefunction with given ℓ

$$R_{k,\ell}(r) = \begin{cases} A j_\ell(kr) & r<R \\ B[j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell] & R<r<\infty \end{cases}$$

Continuity: $A j_\ell(kR) = B[j_\ell(kR)\cos\delta_\ell - n_\ell(kR)\sin\delta_\ell]$

$$\frac{A}{B} = \frac{j_\ell(kR)}{j_\ell(kR)\cos\delta_\ell - n_\ell(kR)\sin\delta_\ell}$$

Derivatives

$$R'_{k\ell}(R+0) - R'_{k\ell}(R-0) = -\lambda R_{k\ell}(R)$$

$$B[j_\ell'(kR)\cos\delta_\ell - n_\ell'(kR)\sin\delta_\ell] - A j_\ell'(kR) = -\lambda A j_\ell(kR)$$

$$\tan\delta_\ell = \frac{\lambda j_\ell^2(kR)}{j_\ell(kR)n_\ell'(kR) - n_\ell(kR)j_\ell'(kR) + \lambda n_\ell(kR)j_\ell(kR)}$$

A3 $\ell = 0$ $\quad j_0(kR) = \frac{\sin kR}{kR}$ $\quad j_0'(\overset{\rho}{kR}) = k\frac{-\sin\rho + \rho\cos\rho}{\rho^2}$

$\quad n_0(kR) = \frac{\cos kR}{kR}$ $\quad n_0'(\overset{\rho}{kR}) = k\frac{-\cos\rho - \rho\sin\rho}{\rho^2}$

$$\tan\delta_0 = \frac{\rho\cdot\lambda\sin^2\rho}{k\sin\rho(-\cos\rho - \rho\sin\rho) - k\cos\rho(-\sin\rho + \rho\cos\rho) + \lambda\rho\sin\rho\cos\rho}$$

$$= \frac{\rho\lambda\sin^2\rho}{-\rho k + \lambda\rho\sin\rho\cos\rho} = \frac{\lambda\sin^2 kR}{-k - \lambda\sin kR\cos kR}$$

resonance $\frac{1}{2}\sin 2kR = {}^1/_{\lambda k}$

bound state

Scattering length

$$a_s = \lim_{kR\to 0} \frac{\lambda \sin^2 kR}{k(k-\lambda \sin kR \cos kR)} \approx \lim_{kR\to 0} \frac{\lambda (kR)^2}{\frac{(kR)^2}{R^2} - \frac{\lambda (kR)^2}{R}}$$

$$= \frac{\lambda R^2}{1-\lambda R}$$

A2 $V(r) = \begin{cases} -U_0 & r<R \\ 0 & r>R \end{cases}$

$$R_\ell(r) = \begin{cases} A j_\ell(k_1 r) & r<R \quad k_1 = \sqrt{\frac{2m(E+U_0)}{\hbar^2}} \\ B[j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell] & r>R \quad k = \sqrt{\frac{2mE}{\hbar^2}} \end{cases}$$

From the class notes

$$\tan\delta_\ell = \frac{kR\, j_\ell'(kR) - \beta_\ell j_\ell(kR)}{kR\, n_\ell'(kR) - \beta_\ell n_\ell(kR)}$$

$$\beta_\ell = \frac{kR}{j_\ell(k_1 R)} \frac{d j_\ell(kR)}{d(kR)}$$

for $\ell = 0$ $\quad \beta_0 = \frac{(k_1R)^2}{\sin^2(k_1R)} \frac{\sin k_1R - (k_1R)\cos k_1R}{(k_1R)^2}$

$$= \frac{1}{\sin k_1R}\left(1 - \frac{k_1R}{\tan k_1R}\right)$$

$$\tan\delta_0 = \frac{k(\sin\rho - \rho\cos\rho) - \beta_0 \rho\sin\rho}{-k(\cos\rho + \rho\sin\rho) - \beta_0\rho\cos\rho}$$

$$a_s = -\lim_{kR\to 0} \frac{\tan\delta_0}{k} = -R\lim_{\rho\to 0} \frac{\frac{1}{R}\left[\rho - \frac{\rho^3}{6} - \rho + \frac{\rho^3}{2}\right] - \beta_0\rho}{-\frac{\rho}{R}\left[1+\rho^2\right] - \beta_0\rho} \quad \sim \rho^4$$

$$= +\lim_{\rho\to 0} \frac{R\beta_0\rho}{\rho\left[\frac{1}{R}+\beta_0\right]} = R\,\frac{R\beta_0}{1+\beta_0 R}$$

A4 Square lattice

In Born approximation

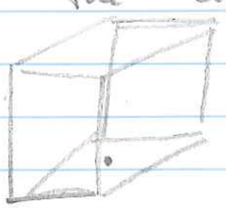
$$f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' e^{i\vec{q}\vec{r}\,'} \sum_i V_0(|\vec{r}\,'-\vec{r}_i|) =$$

$$= -\frac{m}{2\pi\hbar^2}\sum_i e^{i\vec{q}\vec{r}_i}\int d^3\vec{r}\,' e^{i\vec{q}(\vec{r}\,'-\vec{r}_i)} V_0(|\vec{r}\,'-\vec{r}_i|)$$

Fourier transform $= (\sqrt{2\pi})^3 V_0(q)$

$$= -\frac{m\sqrt{2\pi}}{\hbar^2}\sum_i V_0(q)\, e^{i\vec{q}\vec{r}_i} = -\frac{m\sqrt{2\pi}}{\hbar^2} V_0(q)\sum_i e^{i\vec{q}\vec{r}_i}$$

For the cube — center of coordinates are in the center of the cube



$$e^{iq_i L/2} + e^{-iq_i L/2} = 2\cos\frac{q_i L}{2}$$

so

$$f(\vec{q}) = -8\sqrt{2\pi}\,\frac{m V_0(q)}{\hbar^2}\cos\frac{q_x L}{2}\cos\frac{q_y L}{2}\cos\frac{q_z L}{2}$$

For the infinite lattice

$$\sum_{-\infty}^{+\infty} e^{iq_i(nL+\frac{L}{2})} = \frac{e^{iq_i L/2}}{1-e^{-iq_i L}} = \frac{1}{e^{iq_i L/2}-e^{-iq_i L/2}}$$

$$= \frac{2}{\sin q_i L/2}$$

$$f(q) = -8\sqrt{2\pi}\,\frac{mV_0(q)}{\hbar^2}\,\frac{1}{\sin\frac{q_x L}{2}\sin\frac{q_y L}{2}\sin\frac{q_z L}{2}}$$

maxima $\{q_i\}_n = \frac{2\pi n}{L}$

used in X-ray scattering, e- and n- scattering to figure out the crystal structure