Problem 1 (20 points)

Use power series to evaluate the function at the given point:

$\ln(\cos x)+\frac{x^2}{2}$ at $x=0.01$

Useful basic Taylor series

$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}-\ldots$

$\ln(1+x) = x-\frac{x^2}{2}+\ldots$

$$\ln(\cos x) = \ln\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\ldots\right) = \ln\left(1+\left[-\frac{x^2}{2}+\frac{x^4}{24}\right]\right) \approx$$

$$\approx -\frac{x^2}{2}+\frac{x^4}{24}-\frac{1}{2}\left[-\frac{x^2}{2}+\frac{x^4}{24}\right]^2 = -\frac{x^2}{2}+\frac{x^2}{24}-\frac{x^4}{8} = -\frac{x^2}{2}-\frac{x^4}{12}$$

$$\ln(\cos x)+\frac{x^2}{2} \approx \underline{-\frac{x^4}{12}} \approx -\frac{10^8}{12} = \underline{8.33\cdot 10^{-10}}$$

Problem 2 (30 points)

a) (15 points) Simplify:

$$\exp\left[-\frac{\pi}{\substack{\cancel{\sqrt{2}}\\ 2}}\left(\frac{\sqrt{1+i\sqrt{3}}}{\sqrt{2}-i\sqrt{6}}\right)\right]$$

(*in case you don't remember -* $\tan^{-1}\left(\sqrt{3}\right)=\pi/3$)

b) (15 points) Calculate: Arccos (-5/4) (*find all possible answers*)

a) $1+i\sqrt{3} = 2\cdot\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2e^{i\pi/3}$

$\sqrt{2}-i\sqrt{6} = 2\sqrt{2}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = 2\sqrt{2}\,e^{-i\pi/3}$

$$\frac{\sqrt{1+i\sqrt{3}}}{\sqrt{2}-i\sqrt{6}} = \frac{\pm\cancel{\sqrt{2}}\,e^{i\pi/6}}{2\cancel{\sqrt{2}}\,e^{-i\pi/3}} = \pm\frac{1}{2}e^{i\pi/2} = \pm\frac{i}{2}$$

$$\exp\left[\mp\frac{\pi}{2}\cdot\frac{i}{2}\right] = e^{\mp i\pi/4} = \underline{\underline{\frac{1}{\sqrt{2}}\mp\frac{i}{\sqrt{2}}}}$$

b) $\text{Arccos}\left(-\frac{5}{4}\right) = z \Rightarrow \cos z = -\frac{5}{4}$

$\frac{1}{2}\left(e^{iz}+e^{-iz}\right) = -\frac{5}{4} \qquad e^{iz}+e^{-iz}+\frac{5}{2} = 0 \qquad \times e^{iz}$

$\left(e^{iz}\right)^2 + \frac{5}{2}e^{iz} + 1 = 0 \qquad e^{iz} = w$

$w^2 + \frac{5}{2}w + 1 = 0$

$w = -\frac{5}{4} \pm \sqrt{\frac{25}{16}-1} = -\frac{5}{4} \pm \frac{3}{4} = -2, -\frac{1}{2}$

$e^{iz_1} = -2\ ; \quad iz_1 = \text{Ln}(-2) = \text{Ln}\left(2e^{i\pi}\right) = \ln 2 + i\pi + 2\pi i n \quad n=0,\pm1,\pm$

$\underline{z_1 = -i\ln 2 + \pi + 2\pi n} \qquad n = 0, \pm1, \pm2, \ldots$

$e^{iz_2} = -\frac{1}{2} \qquad iz_2 = \text{Ln}\left(-\frac{1}{2}\right) = \text{Ln}\left(\frac{1}{2}e^{i\pi}\right) = -\ln 2 + i\pi + 2\pi i n$

$\underline{z_2 = i\ln 2 - \pi + 2\pi n} \qquad n = 0, \pm1, \pm2, \ldots$

Problem 3 (15 points)

Express the following integral as a B function and then as a combination of Γ functions:

$$\int_0^1 x^{1/6}(1-\sqrt[3]{x})^{3/2}\,dx.$$

Find the exact answer for this integral using $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

$$\int_0^1 x^{1/6}(1-\sqrt[3]{x})^{3/2}\,dx = \left\{ \begin{array}{l} t=\sqrt[3]{x} \\ dt=\frac{1}{3}x^{-2/3}dx \end{array} \right\} = 3\int_0^1 \underbrace{x^{1/6}\cdot x^{2/3}}_{x^{5/6}}(1-x^{1/3})^{3/2}\left[\frac{1}{3}x^{-2/3}dx\right] =$$

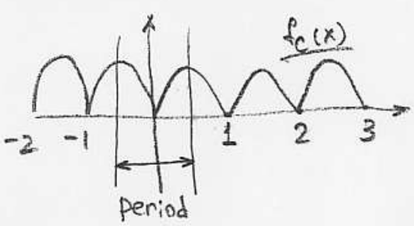
$$= 3\int_0^1 t^{5/2}(1-t)^{3/2}\,dt = 3\,B\left(\frac{7}{2},\frac{5}{2}\right) = 3\,\frac{\Gamma(\frac{7}{2})\Gamma(\frac{5}{2})}{\Gamma(6)} =$$

$$= 3\cdot\frac{\frac{\cancel{5}}{2}\cdot\frac{\cancel{3}}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{\cancel{5}\cdot 4\cdot\cancel{3}\cdot 2\cdot 1} = \frac{9}{2^8}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{9\pi}{256}$$

Problem 4 (35 points)

Given $f(x)=x-x^2$ for $0<x<1$, sketch the even periodic function $f_c(x)$ and the odd periodic functions $f_s(x)$ of period 2, each of which equals $f(x)$ on $0<x<1$. Expand $f_c(x)$ in a cosine series, and $f_s(x)$ in a sine series.

Even function



Because of the symmetry of the function the period ℓ of cos expansion is $2\ell=1$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{\pi n x}{\ell}\,dx = 4\int_0^{1/2}(x-x^2)\cos 2\pi n x\,dx$$

Using given integrals: $\int x\cos kx\,dx = \frac{1}{k^2}\int (kx)\cos kx\,d(kx) = \frac{1}{k}x\sin kx + \frac{1}{k^2}\cos kx$

$$\int x^2\cos kx\,dx = \frac{1}{k^3}\int (kx)^2\cos kx\,d(kx) = \frac{1}{k}x^2\sin kx + \frac{2}{k^2}x\cos kx - \frac{2}{k^3}\sin kx$$

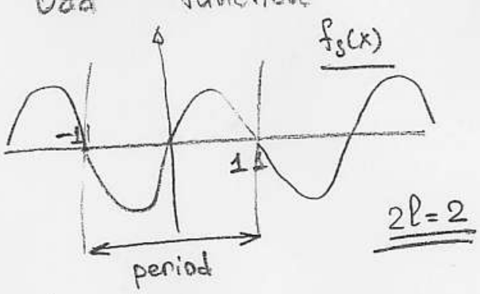
$$a_n = 4\left[\int_0^{1/2} x\cos 2\pi n x\,dx - \int_0^{1/2} x^2\cos 2\pi n x\,dx\right] = 4\left[\overset{=0}{\cancel{\frac{1}{2\pi n}x\sin 2\pi n x\Big|_0^{1/2}}} + \frac{1}{4\pi^2n^2}\cos 2\pi n x\Big|_0^{1/2} -\right.$$

$$\left. - \overset{=0}{\cancel{\frac{1}{2\pi n}x^2\sin 2\pi n x\Big|_0^{1/2}}} - \frac{2}{4\pi^2n^2}x\cos 2\pi n x\Big|_0^{1/2} + \overset{=0}{\cancel{\frac{2}{(2\pi n)^3}\sin 2\pi n x\Big|_0^{1/2}}}\right] = \frac{1}{\pi^2n^2}\left[\cancel{(-1)^n} - 1 - \cancel{(-1)^n}\right] = -\frac{1}{\pi^2 n^2}$$

$$a_0 = 4\int_0^{1/2}(x-x^2)\,dx = 4\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^{1/2} = \frac{1}{3}$$

$$f_c(x) = \frac{1}{6} - \sum_{n=1}^{\infty}\frac{1}{\pi^2n^2}\cos 2\pi n x$$

Odd function



$\underline{2\ell=2}$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{\pi n x}{\ell}\,dx = 2\int_0^1 (x-x^2)\sin \pi n x\,dx =$$

$$= 2\left[\int_0^1 x\sin\pi n x\,dx - \int_0^1 x^2\sin\pi n x\,dx\right] = \left\{\begin{matrix}\text{using}\\ \text{provided}\\ \text{integrals}\end{matrix}\right\} =$$

$$= 2\left[-\frac{1}{\pi n}x\cos\pi n x\Big|_0^1 + \overset{=0}{\cancel{\frac{1}{\pi^2n^2}\sin\pi n x\Big|_0^1}} + \frac{1}{\pi n}x^2\cos\pi n x\Big|_0^1 + \right.$$

$$\left. - \overset{=0}{\cancel{\frac{2}{\pi^2n^2}x^2\sin\pi n x\Big|_0^1}} - \frac{2}{\pi^3n^3}\cos\pi n x\Big|_0^1\right] = 2\left[-\cancel{\frac{(-1)^n}{\pi n}} + \cancel{\frac{(-1)^n}{\pi n}} - \frac{2[(-1)^n-}{\pi^3 n^3}\right.$$

$$= \begin{cases}\dfrac{8}{\pi^3n^3} & n \text{ is odd}\\ 0 & n \text{ is even}\end{cases}$$

$$f_s(x) = \sum_{n=1}^{\infty}\frac{8}{\pi^2n^2}\sin\pi n x = \sum_{k=0}^{\infty}\frac{8}{\pi^2(2k+1)^2}\sin\pi(2k+1)x$$