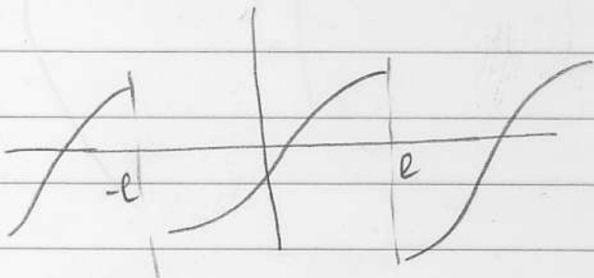
Lecture # 7
2009

Fourier transforms



For a periodic function (with a period 2ℓ)

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{\frac{i\pi n x}{\ell}}$$

wher $c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(y) \, e^{-\frac{i\pi n y}{\ell}} dy$

The Fourier series is a sum of descrete harmonics $\omega_n = \frac{\pi n}{\ell}$, $\Delta\omega = \frac{\pi}{\ell}$. As ℓ increases, the separation between separate harmonics $\Delta\omega$ becomes smaller

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \, e^{i\frac{\pi n x}{\ell}} \underbrace{\Delta n}_{=1} = \frac{\ell}{\pi} \sum_{n=-\infty}^{+\infty} c_n \, e^{i\omega_n x} \Delta\omega =$$

$$= \frac{\cancel{\ell}}{\pi} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2\cancel{\ell}} \int_{-\ell}^{\ell} f(y) \, e^{-i\omega_n y} dy \right) e^{i\omega_n x} \Delta\omega =$$

$$= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\ell}^{\ell} f(y) \, e^{-i\omega_n y} dy \right) e^{i\omega_n x} \Delta\omega \quad \xRightarrow{\ell \to \infty} \quad \Delta\omega = \pi/\ell \to 0, \quad \omega_n \to \omega, \quad \sum_{n=-\infty}^{+\infty} \dots \Delta\omega \to \int_{-\infty}^{+\infty} \dots d\omega$$

$$= \int_{-\infty}^{+\infty} \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(y) \, e^{-i\omega y} dy \right]}_{g(\omega) \text{ — Fourier transform}} e^{i\omega x} d\omega$$

$$f(x) = \int_{-\infty}^{+\infty} g(\omega) \, e^{i\omega x} d\omega \; ; \quad g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(y) \, e^{-i\omega y} dy$$

Sometimes the factor $\frac{1}{2\pi}$ is equally divided b/w two expressions

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega) \, e^{i\omega x} dx \; ; \quad g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \, e^{-i\omega y} dy$$

Orthogonality condition

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y) \, e^{-i\omega y} dy \, e^{i\omega x} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \cdot f(y) \int_{-\infty}^{+\infty} e^{i\omega (x-y)} d\omega$$

should be $= \delta(x-y)$ – Dirac δ-function

Properties of a delta function

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

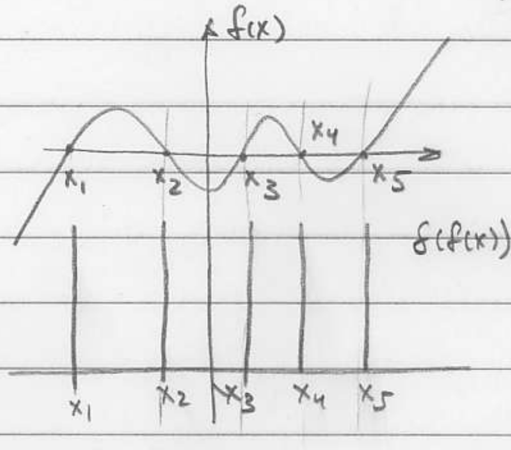
$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \qquad \int_a^b f(x)\,\delta(x-x_0)\,dx = \begin{cases} f(x_0) & x_0 \in [a,b] \\ 0 & x_0 \notin [a,b] \end{cases}$$

Integral form of the Delta function

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x}\,d\omega \qquad \delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(x-x_0)}\,d\omega$$

Properties

1. $\delta(x) = \delta(-x)$
2. $\delta(ax) = \frac{1}{|a|}\,\delta(x)$ $\qquad \delta(ax) = \delta(|a|x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega|a|x}\,d\omega = \frac{1}{|a|}\delta(x)$
3. $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$
4. $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ $\qquad$ where $f(x_i) = 0$, x_i – zeros of $f(x)$



Around each zero x_i

$$f(x \approx x_i) = f'(x_i)\,(x - x_i)$$

$$\delta(f(x)) \approx \delta(f'(x_i)(x-x_i)) = \frac{1}{|f'(x)|}\,\delta(x-x_i)$$

5. Derivative of Delta function $\delta'(x)$

It worth thinking about it in the context of an integral

$$\int_a^b f(x)\,\delta'(x-x_0)\,dx = \underbrace{\cancel{f(x)\,\delta(x-x_0)\Big|_a^b}}_{=0} - \int_a^b f'(x)\,\delta(x-x_0)\,dx$$

$$\int_a^b f(x)\,\delta'(x-x_0)\,dx = -\int_a^b f'(x)\,\delta(x-x_0)\,dx$$

6. Integral $\int_{-\infty}^{x} \delta(y)\,dy = \Theta(x)$ — Step function

Three - dimentional Fourier transform

$f(x) \longrightarrow f(x,y,z) = f(\vec{r})$ $\qquad e^{i\omega x} \longrightarrow e^{ik_x x}, e^{ik_y y}, e^{ik_z z} = = e^{i\vec{k}\vec{r}}$

$$f(\vec{r}) = \int g(\vec{k})\, e^{i\vec{k}\vec{r}}\, d^3\vec{r} = \int g(k_x, k_y, k_z)\, e^{ik_x \cdot x}\, e^{ik_y \cdot y}\, e^{ik_z \cdot z}\, dx\,dy\,dz$$

$$g(\vec{k}) = \frac{1}{(2\pi)^3} \int f(\vec{r})\, e^{-i\vec{k}\vec{r}}\, d^3\vec{k} = \int f(x,y,z)\, e^{-ik_x \cdot x}\, e^{-ik_y \cdot y}\, e^{-ik_z \cdot z}\, dk_x\,dk_y\,dk_z$$

Again, sometimes $f(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\vec{k})\ldots$, and $g(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{r})\ldots$

Sine and Cosine Fourier transform

For an even function $f(x) = f(-x)$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \Big[\underbrace{\int_{-\infty}^{0} f(x)\, e^{-i\omega x}\, dx}_{x \to -x} + \int_{0}^{\infty} f(x)\, e^{-i\omega x}\, dx \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(x)\, (e^{i\omega x} + e^{-i\omega x})\, dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x\, dx$$

If $f(x)$ is even, $g(\omega)$ is also even $(g(\omega) = g(-\omega))$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(\omega) \cos \omega x\, d\omega$$

Similarly for an odd function $f(x) = -f(x)$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \Big[\int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx + \int_0^{\infty} f(x)\, e^{-i\omega x}\, dx \Big]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(x) (e^{-i\omega x} - e^{i\omega x})\, dx = \sqrt{\frac{2}{\pi}} \cancel{\left(\frac{1}{i}\right)} \int_0^{\infty} f(x) \sin \omega x\, dx$$

remove, since it will cancell out with "i" in the expression for $g(\omega)$

$$g(\omega) = -g(-\omega) \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(\omega) \sin \omega x\, d\omega$$

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x\, dx$$

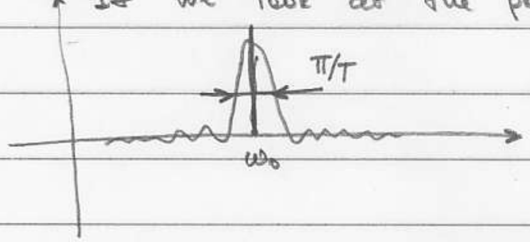
An example: time - limited plain wave

1. True plane wave: $E(t) = E_0 \cos\omega_0 t \qquad -\infty < t < +\infty$

Its spectrum: $E(\omega) = \frac{1}{2\pi} E_0 \int \cos\omega_0 t \; e^{-i\omega t}\, dt =$

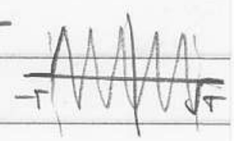
$$= \frac{1}{4\pi} E_0 \int \left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{-i\omega t}\, dt = \frac{1}{2} E_0 \left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right)$$

If we look at the positive part of the spectrum



Time-limited plain wave

$$E(t) = \begin{cases} E_0 \cos\omega_0 t & -T < t < T \\ 0 & |t| > T \end{cases}$$



Spectrum!

Then $E(\omega) = \frac{1}{2\pi} E_0 \int_{-T}^{T} \cos\omega_0 t \; e^{-i\omega t}\, dt = \frac{1}{4\pi} E_0 \int_{-T}^{T} \left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) e^{-i\omega t}\, dt =$

$$= \frac{1}{4\pi} E_0 \left[\frac{e^{i(\omega-\omega_0)t}}{\omega-\omega_0} \Bigg|_{-T}^{T} - \frac{e^{-i(\omega+\omega_0)t}}{\omega+\omega_0} \Bigg|_{-T}^{T} \right] =$$

$$= \frac{1}{2\pi} E_0 \left[\frac{\sin(\omega-\omega_0)T}{\omega-\omega_0} - \cancel{\frac{\sin(\omega+\omega_0)T}{\omega+\omega_0}} \right] \qquad \text{if } \omega, \omega_0 - \text{large}$$

$\sim \frac{1}{\omega_0}$, small

~~Hto~~ Peaks at $\omega = \omega_0$, hight of the peak $\sim T$

width $\quad (\omega-\omega_0)T = \pi/2 \qquad \omega - \omega_0 = \frac{\pi}{2T}$

Properties of the Fourier transforms

1. Complex congugation: if $f(x)$ is real $\Rightarrow g^*(\omega) = g(-\omega)$

$$g^*(\omega) = \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx\right)^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx = g(-\omega)$$

2. Differentiation:

$f(x) \rightarrow g(\omega)$; what is the Fourier transform for $f'(\omega)$?

$$g_1(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f'(x)\, e^{-i\omega x}\, dx = \cancel{\frac{1}{2\pi} f(x)\, e^{-i\omega x}\Big|_{-\infty}^{+\infty}}^{=0} - \frac{1}{2\pi}\underline{\int_{-\infty}^{+\infty} (-i\omega) f(x)\, e^{-i\omega x} dx}$$

$\left(f(x) \xrightarrow{x\to\infty} 0 \right)$

$$g_1(\omega) = (i\omega)\, g(\omega)$$

In general; fourier transform of $f^{(n)}(x)$ is $g_{(n)}(\omega) = (i\omega)^n g(\omega)$

3. Attenuation

$f_a(x) \longrightarrow e^{-ax} f(x)$ $\qquad g_a(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{-ax}\, e^{-i\omega x}\, dx = g_0(\omega - ia)$

4. Coordinate shift

$f_a(x) = f(x-a)$ $\qquad g_a(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-a)\, e^{-i\omega(x-a)}\, e^{-i\omega a}\, dx =$

$$= e^{-i\omega a}\, g(\omega)$$

5. Convolution

Definition of a convolution integral:

$$C(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dx$$

Fourier transform for the convolusion

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy\right) e^{-i\omega x}\, dx = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} dx\, dy\, \left(g(x-y)\, e^{-i\omega(x-y)}\, dx\right) \times$$

$$\times\ f(y)\, e^{-i\omega y}\, dy = \underline{F(\omega)\ G(\omega)}$$

A fourier transform of a convolution is a product of two FTs.

Persival theorem for Fourier transforms

For the descrete series $f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} \Rightarrow \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |c_n|^2$

For the transform we define similar average $|f(x)|^2$

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} f(x) f^*(x) dx = \int_{-\infty}^{+\infty} dx \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega_1) e^{i\omega_1 x} d\omega_1 \right) \left(\frac{1}{\sqrt{2\pi}} g^*(\omega_2) e^{-i\omega_2 x} d\omega_2 \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega_1 d\omega_2 \, g(\omega_1) g^*(\omega_2) \int_{-\infty}^{+\infty} e^{i(\omega_1 - \omega_2)x} dx = \int_{-\infty}^{+\infty} d\omega_1 d\omega_2 \, g(\omega_1) g^*(\omega_2) \delta(\omega_1 - \omega_2) =$$

$$= \int_{-\infty}^{+\infty} |g(\omega)|^2 d\omega$$