Lecture # 6
2009

Fourier decomposition $\longleftrightarrow$ vector decomposition analog

Vectors

$\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ - complete basis

$\vec{A} = A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$

Orthogonality

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \qquad i,j = x,y,z$$

Vector components (using orthogonality)

$$A_i = \vec{A}\,\vec{e}_i$$

Fourier series

$\{\sin nx, \cos nx\}$ - complete basis

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Orthogonality

$$\int_{-\pi}^{\pi} \sin nx \cos mx = 0 \quad \text{for any } n,m$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \delta_{nm}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \begin{cases} \delta_{nm} & n,m \neq 0 \\ 2 & n=m=0 \end{cases}$$

Coefficient of expansion

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Complex Fourier series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

Orthogonality $\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \delta_{nm}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$$

So far we assumed $f(x)$ is periodic at the interval $[-\pi, \pi]$.
For any different period $[-\ell, \ell]$: $\quad x \longrightarrow \frac{\pi x}{\ell}$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \cos \frac{\pi n x}{\ell}\,dx \quad ; \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x) \sin \frac{\pi n x}{\ell}\,dx$$

Cosine and Sine Fourier series

If $f(x)$ is symmetric $\to$ $b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \overbrace{\int_{-\pi}^{0} f(x)\cos nx\,dx}^{x \to -x}\right] =$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad \left[\text{or} \quad \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{\pi n x}{\ell}\,dx\right]$$

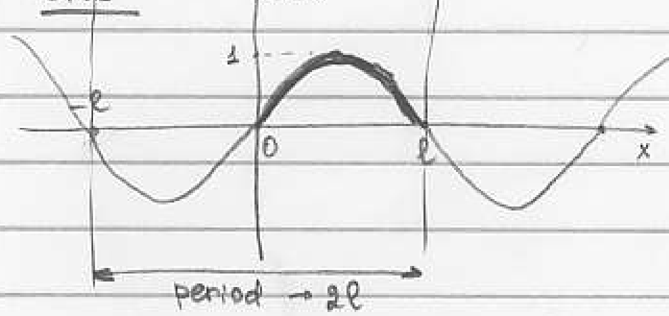
If $f(x)$ is anti-symmetric $\longrightarrow$ $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad \left[\text{or} \quad \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{\pi n x}{\ell}\,dx\right]$$

Function decomposition into sine and cosine series. Often function is defined only inside certain interval, and the symmetry of the problem will dictate if it should be decomposed into sine or cosine series.

Example:

Sine

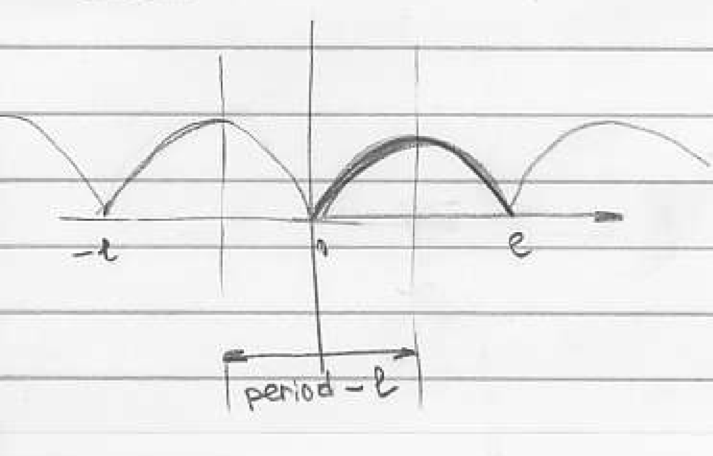


$$f(x) = \frac{4x(\ell - x)}{\ell^2}$$

Sine series $\to$ continue the function as periodic anti-symmetric.

$$b_n = \frac{2}{\ell}\int_0^{\ell} \frac{4x(\ell-x)}{\ell^2}\sin\frac{\pi n x}{\ell}\,dx$$

Cosine



Cosine series $\to$ continue the function as periodic symmetric function. Because of the symmetry of the function, period is ℓ, not 2ℓ

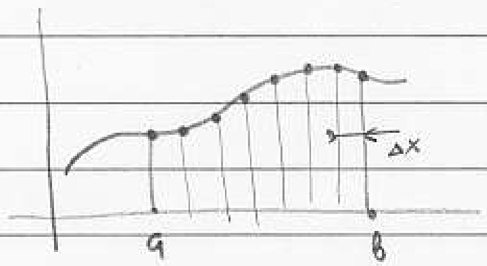
$$a_n = \frac{4}{\ell}\int_0^{\ell/2} f(x)\cos\frac{2\pi n x}{\ell}\,dx$$

Average of a function – definition

It is easy to find the average of a set of numbers

$$\langle n \rangle = \frac{n_1 + n_2 + \ldots + n_N}{N}$$

Let's approach functions similaly, breaking them into N descrete points, and averaging these values



$$\langle f(x) \rangle_N = \frac{f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots + f(a+(N-1)x) \times \Delta x}{N \cdot \Delta x}$$

$$\Delta x = \frac{b-a}{N} \quad \Rightarrow \quad N \cdot \Delta x = (b-a)$$

$$\langle f(x) \rangle = \lim_{\substack{N \to \infty \\ \text{or } \Delta x \to 0}} \frac{\sum_{n=0}^{N-1} f(a + n \cdot \Delta x) \cdot \Delta x}{b-a} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Average of $\sin x$ and $\cos x$ over the period

$$\int_{-\pi}^{\pi} \sin x \, dx = \int_{-\pi}^{\pi} \cos x \, dx = 0 \quad , \text{ so } \quad \langle f(x) \rangle = \frac{1}{2} a_0$$

<u>Persival</u> <u>theorem</u> provides a relation between average of the square of a function and its Fourier series coefficients

If $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, then

$$\langle f(x)^2 \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 + \frac{1}{2} \sum_{n=1}^{\infty} b_n^2$$

Using Fourier series

$$f^2(x) = \left[\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right] \times \left[\frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \right]$$

$$= \frac{1}{4} a_0^2 + a_0 \left[\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \cos nx \right] + 2 \sum_{n,k=1}^{\infty} a_n b_k \cos nx \sin kx +$$

$$+ \sum_{n,k=1}^{\infty} a_n a_k \cos nx \cos kx + \sum_{n,k=1}^{\infty} b_n b_k \sin nx \sin kx$$

$$\langle f^2(x)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{4}a_0^2 + \sum_{n,k=1}^{\infty} a_n a_k \cdot \frac{1}{2}\delta_{nk} + \sum_{n,k=1}^{\infty} b_n b_k \frac{1}{2}\delta_{nk} =$$

$$= \underline{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n^2}$$

Persival theorem for a complex function

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} \Rightarrow \langle |f(x)|^2\rangle = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$|f(x)|^2 = f(x)\cdot f^*(x) = \sum_{n,k=-\infty}^{+\infty} c_n c_k^* \, e^{inx} e^{-ikx}$$

$$\langle |f(x)|^2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n,k=-\infty}^{+\infty} c_n c_k^* e^{inx} e^{-ikx}\,dx = \sum_{n,k=-\infty}^{+\infty} c_n c_k^* \delta_{nk} = \sum_{n=-\infty}^{+\infty} |c_n|^2$$

Convergence of Fourier series: Dirichlet condition.

If a function $f(x)$ is: periodic at the interval $[-\pi, \pi]$

$f(x)$ is piecewise regular $\left\{\begin{array}{l}\text{single valued} \\ \text{has finite number of minima \& maxima} \\ \text{has finite number of discontinuities} \\ \text{and } \int_{-\pi}^{\pi} |f(x)|\,dx \text{ is finite}\end{array}\right.$

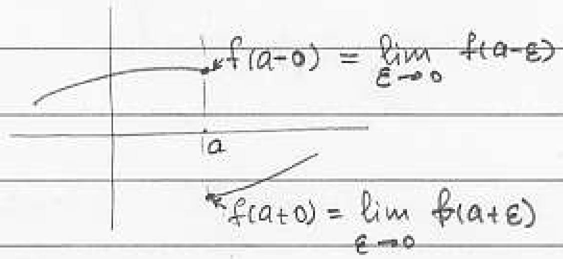
then the corresponding Fourier converges to its function at all points where $f(x)$ is continous; at discontinuities the Fourier series converges to the midpoint of the jump.

Dirichlet condition is sufficient, but not necessary (i.e there exist functions that have converging Fourier series, but violate Dirichlet condition → for example $\sin\frac{1}{x}$)

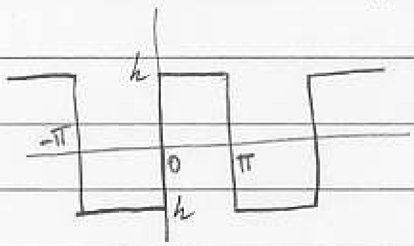
Behavior at Discontinuities

From the Dirichler condition the Fourier series converges to

$$\frac{1}{2}\left(f(a+0) + f(a-0) \right)$$



Example: square wave.



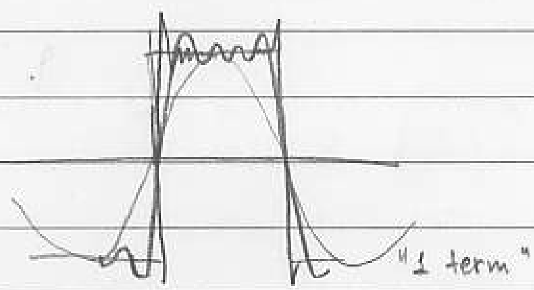
$$f(x) = \begin{cases} h & 0 < x < \pi \\ -h & -\pi < x < 0 \end{cases}$$ odd function

can expand into sine series

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = -\frac{2}{\pi n} \cos nx \Big|_0^{\pi} =$$

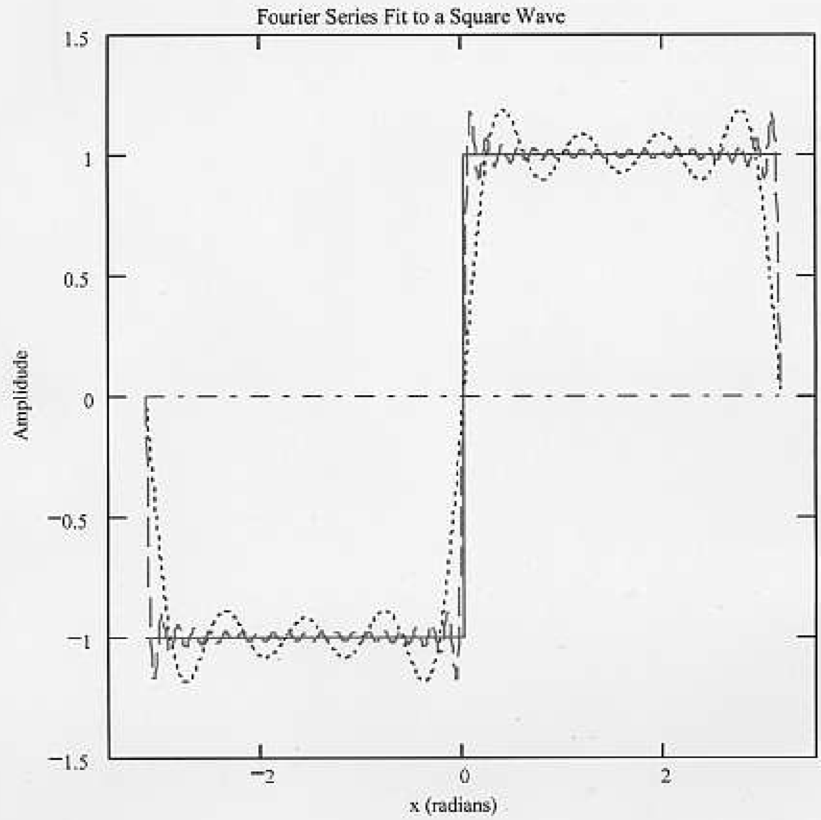
$$= -\frac{2}{\pi n} \underbrace{\left[(-1)^n - 1 \right]}_{= \begin{cases} 0 & n = 2k \\ -2 & n = 2k+1 \end{cases}} = \frac{4}{\pi(2k+1)} \qquad k = 0, 1, 2, \ldots$$

$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$$

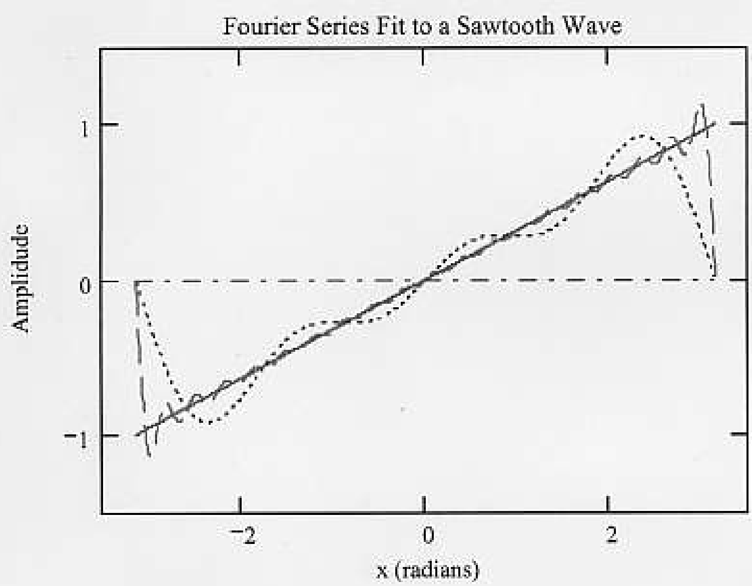


Accuracy of the function representation as the number of included terms increases:

1. There is a steady increase in the accuracy of representation (although Fourier series tend to converge slower than Taylor series)
2. All the curve pass through the midpoints at the ends
3. Near the jump there is an overshoot of $\sim 10\%$. It persists and gets closer to the point of discontinuity as more and more terms are included.



blue – 4 terms
green – 20 terms



Integration and differentiation of the series

Term-by-term integration of a Fourier series generally leads to the converging Fourier series.

Indeed

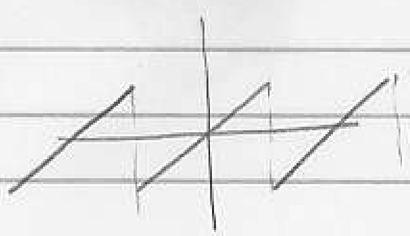
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\int f(x)\,dx = \underline{\underline{\frac{1}{2}a_0 x}} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos nx$$

Integration places an additional "n" in the denominator of all coefficients $\longrightarrow$ we can expect the series to converge faster.

<u>However</u>, term-by-term <u>differentiation</u> may not work

Example: $f(x) = x \qquad -\pi < x < \pi$



odd function $\longrightarrow$ sine series

$$b_n(x) = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = -\frac{2}{\pi n}\, x\cos nx \Big|_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi} \cos nx\,dx$$

$$= -\frac{2}{\pi n}\,\pi \cos \pi n = 2(-1)^{n+1}\frac{1}{n}$$

$$f(x) = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

$$\int_0^x f(x)\,dx = 2\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}\Big|_0^x =$$

$$= 2\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} + 2\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2} = \frac{x^2}{2}$$

However: $f'(x) = 1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$ $\longrightarrow$ does not converges.