Lecture # 3
2009

It is very common in physics to use a summation to calculate a certain value, or to sum various contributions to calculate the total effect.

Very common summations

Sum $S_N = a_0 + a_2 + \ldots + a_N = \sum_{n=0}^{N} a_n$ finite calculatable sum.

Arithmetic progressions $a_n = a + b \cdot n$

$$S_N = \sum_{n=0}^{N} a_n = \frac{(N+1)}{2}(a_0 + a_N) = (N+1)\left(a + \frac{bN}{2}\right)$$

Geometrical progression $a_n = a \cdot \alpha^n$

$$S_N = \sum_{n=0}^{N} a_n = a\,\frac{\alpha^{N+1} - 1}{\alpha - 1}$$

Proof: $S_N = a + \alpha a + \ldots \alpha^N a \Rightarrow \alpha S_N = \alpha a + \alpha^2 a + \ldots \alpha^{N+1} a = S_N - a + \alpha^{N+1} a$

$$(\alpha - 1) S_N = a(\alpha^{N+1} - 1)$$

$$S_N = a\,\frac{\alpha^{N+1} - 1}{\alpha - 1}$$

Binomial sum

$$S_N = (1+x)^N = \sum_{n=0}^{\infty} \binom{N}{n} x^n \qquad a_n = \binom{N}{n} x^n$$

$$\boxed{\binom{N}{n} = \frac{N!}{n!(N-n)!} \quad n! = 1 \cdot 2 \cdot 3 \cdots n! \quad 0! = 1}$$

Infinite series $S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$

Geometrical progression

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} a\,\frac{\alpha^{N+1} - 1}{\alpha - 1} \longrightarrow \begin{cases} \alpha < 1 & S = \frac{a}{\alpha - 1} \\ \alpha > 1 & S_n \to \infty \text{ diverges} \\ \alpha = 1 & S = \sum_{n=0}^{\infty} a = \text{diverges} \end{cases}$$

Guessing the convergence of the following series:

$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\dots$ $\qquad \sum_{n=1}^{\infty} \frac{e^n}{n!} \qquad \sum_{n=1}^{\infty} \frac{(n-\ln n)}{5n^2-3n+1}$

$\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$ D

Tests of convergence

1. Preliminary test: if $\lim_{n\to\infty} a_n \neq 0$ such series diverges

$$a_n = \frac{3^n}{2^n+3^n} \qquad \lim_{n\to\infty} \frac{3^n}{2^n+3^n} = \lim_{n\to\infty} \frac{1}{1+(\frac{2}{3})^n} = 1$$ Series diverges

2. Integral test [Maclauren test]

The series $\sum_{n=1}^{\infty} a_n$ $\binom{\text{converges}}{\text{diverges}}$ if $\int^{\infty} a_n\, dn$ is $\binom{\text{finite}}{\text{infinite}}$ at upper limit

$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow \int^{\infty} \frac{1}{n}\, dn = \ln(n) \Big|^{\infty} = \infty$ diverges at the upper limit

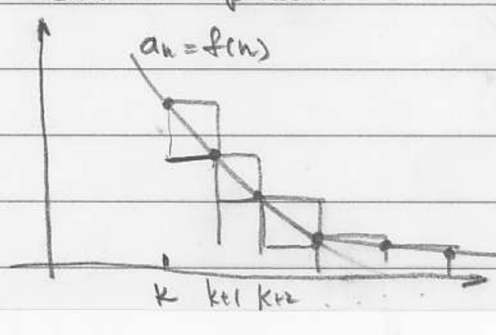
so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \int^{\infty} \frac{1}{n^2}\, dn = \frac{1}{n} \Big|^{\infty} = 0$ converges at the upper limit

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for any $\alpha > 1$ according to the integral test

Short proof



$; \int^{\infty} f(n)\, dn < \sum_{n=k}^{\infty} a_n; \quad \int^{\infty} f(n)\, dn > \sum_{n=k+1}^{\infty} a_n$

$$\int^{\infty} f(n)\, dn < \sum_{n=k}^{\infty} a_n < \int^{\infty} f(n)\, dn + a_k$$

If $\int^{\infty} f(n)\, dn < \infty$, then the summation converges.

3. Ratio test [D'Alembert test]

If $S = \sum\limits_{n=0}^{\infty} a_n$ and $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$, then the series

converges if $\rho < 1$, diverges of $\rho > 1$; this test does not work for $\rho = 1$.

Short proof: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ means that for any small ε we can find $n = N$ such that for any $n > N$ $\left|\frac{a_{n+1}}{a_n} - \rho\right| < \varepsilon$

We can safely assume that for any $n > N$

$\frac{a_{n+1}}{a_n} < (\rho + \varepsilon)$, and $a_{n+1} < (\rho+\varepsilon) a_n$; $a_{n+1} < (\rho+\varepsilon)^{(n-N)} a_N$

Then $\sum\limits_{n=N}^{\infty} a_n < a_N \sum\limits_{n=N}^{\infty} (\rho+\varepsilon)^{n-N} = \frac{a_N}{1-(\rho+\varepsilon)}$ for $\rho+\varepsilon < 1$

$\varepsilon \to 0$, so $\rho < 1$

$\sum \frac{e^n}{n!}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)!} \frac{n!}{e^n} = \frac{e}{n+1} \xrightarrow{n\to\infty} 0$ series converges

This test is particularly useful for the series containing $n!$

4. Comparison test

a) if $\sum\limits_{n=0}^{\infty} b_n$ is converging series of positive terms and $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ is finite, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

b) if $\sum\limits_{n=0}^{\infty} b_n$ is diverging series of positive terms and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} \neq 0$ $(>0, \infty)$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

$a_n = \frac{n - \ln n}{5n^2 - 3n + 1}$ $\Rightarrow$ let's leave only leading terms $\Rightarrow$ $\frac{n}{5n^2} = \frac{1}{5n}$

$b_n = \frac{1}{n}$ – diverges. $\qquad \lim \frac{a_n}{b_n} = \frac{n(n-\ln n)}{5n^2-3n+1} = \frac{1}{5} \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverges

Alteratnating series

$\sum \frac{(-1)^n}{n}$ - converges conditionally; diverges absolutely

Absolute convergence => in $\sum |a_n|$ converges.

Conditional convergence: If absolute values of the terms decrease steadily to zero ($|a_{n+1}| < |a_n|$) and $\lim_{n\to\infty} |a_n| = 0$ then the series converges conditionally.

Series of Functions

$$S(x) = \sum_{n=0}^{\infty} a_n f_n(x) \rightarrow \begin{cases} x^n & \text{(Power series)} \\ [\sin nx, \cos nx] & \text{Fourier series} \\ \text{(special functions \& polynomials} \end{cases}$$

We will talk about power series.

$$S(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for power series we need to talk about interval of convergence

$S(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ converges for $|x| < 1$

$S(x) = \sum_{n=0}^{\infty} \frac{(2x)^n}{3^{2n}}$ → Convenient to use ratio test

$\lim \frac{a_{n+1}}{a_n} = \left|\frac{2x}{9}\right| < 1$ for $|x| < \frac{9}{2}$

$S(x) = \sum_{n=0}^{\infty} \frac{x^n}{\ln n + n!}$ → Convenient to use the comparison test

$b_n = \frac{x^n}{n!}$ → converging by the ratio test for any x

thus $S(x)$ converges for any x

If we talk about complex series, it is more accurate to talk about the disc of convergence

$$\sum_{n=1}^{\infty} \frac{(iz)^n}{4n^2} \quad \text{ratio test} \quad \lim_{n\to\infty} \left| \frac{(iz)^{n+1}}{4(n+1)^2} \frac{n^2}{(iz)^n} \right| = |z| < 1$$

it also converges for $|z| = 1$ by comparison test

Complex series

$\sum_{n=0}^{\infty} c_n$ absolute convergence: if $\sum_{n=0}^{\infty} |c_n|$ converges.

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} (a_n + i b_n) = \underbrace{\sum_{n=1}^{\infty} a_n} + i \underbrace{\sum_{n=1}^{\infty} b_n}$$

If these two series converge independently, $\sum_{n=0}^{\infty} c_n$ converges

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Start Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \; ; \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \; ; \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2} x^2 + \frac{p(p-1)(p-2)}{3!} x^3$$

$$\binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{p(p-1) \quad (p-n+1)}{n!}$$

If p is not integer, the last series is infinite