Examples of complex numbers applications in physics.

Electromagnetic wave

$$\frac{\partial^2 E}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \qquad E_x(z,t) = E_0 \cos(kz - \omega t + \varphi)$$

A word of caution! There are two ways to turn a real $\cos(x)$ function into a complex exponent

① $E_0 \cos(kz-\omega t+\varphi) = \frac{1}{2}E_0\left[e^{i(kz-\omega t+\varphi)} + e^{-i(kz-\omega t+\varphi)}\right] = \underline{\frac{1}{2}E_0}\, e^{i(kz-\omega t+\varphi)} + c.c.$

The amplitude in front of the complex exponent is $\underline{(\frac{1}{2}E_0)}$

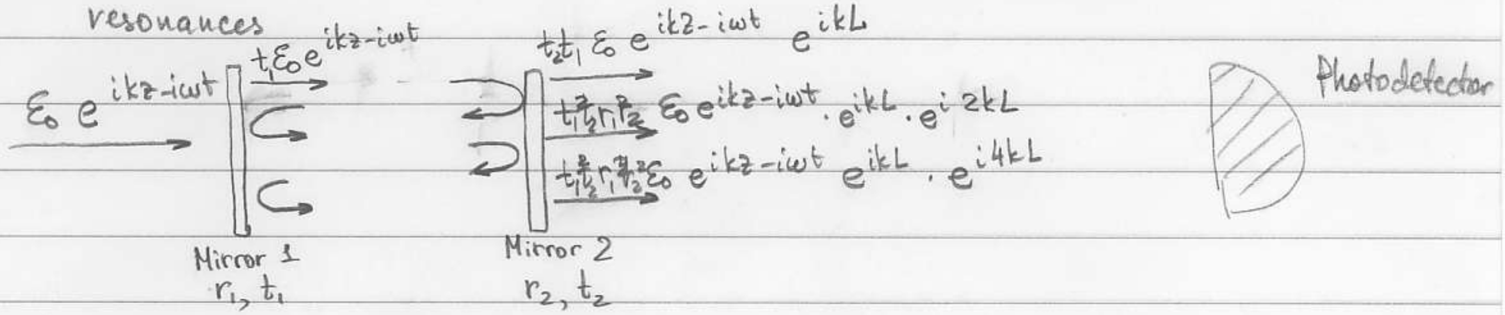
② $E_0 \cos(kz-\omega t+\varphi) = \mathrm{Re}\left[E_0 e^{i(kz-\omega t+\varphi)}\right]$

The amplitude in front of the complex exponent is $(\underline{E_0})$

The definitions of some physical values can be a factor of 2 different because of different conversion ways.

Use $\mathcal{E}_0 e^{i(kz-\omega t+\varphi)}$ for EM wave

Example 1: use this form to evaluate Fabri-Perot interferometer resonances



$$\mathcal{E}_{out}(z,t) = t_1 t_2 \mathcal{E}_0 e^{ikz-i\omega t} e^{ikL}\left[1 + r_1 r_2 e^{i2kL} + r_1^2 r_2^2 e^{i4kL} + \dots\right] =$$

Geometrical series $1 + x + x^2 + \dots = \frac{1}{1-x}$

$$= t_1 t_2 \mathcal{E}_0 e^{ikz-i\omega t} e^{ikL} \frac{1}{1 - r_1 r_2 e^{i2kL}}$$

Photodiodes detect light intensity

$$\text{Signal} \propto |\mathcal{E}_{out}|^2 = t_1^2 t_2^2 \, |\mathcal{E}_0|^2 \frac{1}{|1-r_1 r_2 e^{i2kL}|^2} =$$

$$= |\mathcal{E}_0|^2 \frac{t_1^2 t_2^2}{(1-r_1 r_2 e^{i2kL})(1-r_1 r_2 e^{-i2kL})} = |\mathcal{E}_0|^2 \frac{t_1^2 t_2^2}{1+r_1^2 r_2^2 - r_1 r_2 (e^{i2kL}+e^{-i2kL})} =$$

$$= |\mathcal{E}_0|^2 \frac{t_1^2 t_2^2}{1-2r_1 r_2 \cos 2kL + r_1^2 r_2^2}$$

To make the expression look nicer put $t_1 = t_2 = t$, $r_1 = r_2 = r$

for a perfect mirror

$$\text{Signal} \propto |\mathcal{E}_0|^2 \frac{t^4}{1-2r^2\cos 2kL + r^4} = \frac{(1-r^2)^2}{(1-2r^2\cos 2kL + r^4)} \cdot |\mathcal{E}_0|^2$$

Resonance condition : $\cos 2kL = 1 \Rightarrow 2kL = 2\pi n$ $\quad n = 0, 1, 2 \ldots$

Then everything is transmitted!

To estimate the width of the resonance rewrite

(i.e how much change in length one can detect)

$$\text{Signal} \propto |\mathcal{E}_0|^2 \frac{t^4}{t^4 + 2r^2(1-\cos 2kL)}$$

For HWHM

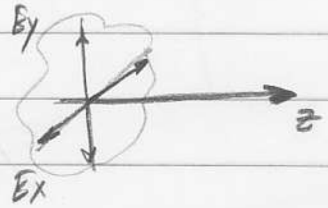
$$\frac{t^4}{t^4 + 2r^2(1-\cos 2k\Delta L)} = \frac{1}{2} \quad \Rightarrow \quad (1-\cos 2kL) = \frac{t^4}{2r^2}$$

For a normal mirror $\quad r^2 \approx 0.99 \quad t^2 = 10^{-2}$

$(1-\cos 2kL) = 5 \cdot 10^{-5}$ $\quad 2k\Delta L \approx 0.01$ $\quad 2 \cdot \frac{2\pi}{\lambda} \cdot \Delta L = 0.01$

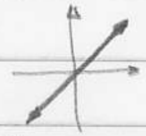
$$\Delta L = \underline{8 \cdot 10^{-4} \lambda}$$

Example 2: Polarization of EM field



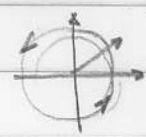
$$\vec{E}(z,t) = (\mathcal{E}_x \vec{e}_x + \mathcal{E}_y \vec{e}_y)\, e^{ikz-i\omega t}$$

Linear polarization



E_x & E_y oscillate in phase

Circular polarization



$$\vec{E}(z,t) = \mathcal{E}_0 \vec{e}_x \cos(kz-\omega t) + \mathcal{E}_0 \vec{e}_y \sin(kz-\omega t)$$

$$= \mathcal{E}_0 \vec{e}_x \,\mathrm{Re}\left[e^{ikz-i\omega t}\right] + \mathcal{E}_0 \vec{e}_y \underbrace{\mathrm{Im}\left[e^{ikz-i\omega t}\right]}_{= \mathrm{Re}\left[\frac{1}{i} e^{ikz-i\omega t}\right]}$$

$$= \mathrm{Re}\left[\mathcal{E}_0\, e^{ikz-i\omega t} \left[\vec{e}_x - i\vec{e}_y\right]\right] =$$

$$= \mathrm{Re}\left[\mathcal{E}_0\, e^{ikz-i\omega t}\, \vec{e}_-\right]$$

Unit vectors for circular polarizations: $\vec{e}_+ = \vec{e}_x + i\vec{e}_y$ "σ_+" polarization

$\vec{e}_- = \vec{e}_x - i\vec{e}_y$ "σ_-" polarization

Example 3: Complex physical values. (EM waves in a conductive medium)

$$\boxed{\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \nabla \vec{D} = 0 \\ \vec{\nabla}\times\vec{H} = \vec{j} + \dfrac{\partial \vec{D}}{\partial t} & \nabla \vec{B} = 0 \end{cases}} \Rightarrow \begin{array}{l} \vec{D} = \varepsilon_0 \varepsilon \vec{E} \\ \vec{H} = \dfrac{1}{\mu\mu_0}\vec{B} \\ \vec{j} = \sigma \cdot \vec{E} \end{array} \quad \begin{array}{l} \vec{E} = \mathcal{E}_0 \vec{e}_x\, e^{ikz-i\omega t} \\ \vec{B} = B_0 \vec{e}_y\, e^{ikz-i\omega t} \\ \sigma - \text{conductivity} \end{array}$$

By substituting the expressions

$$B_0 = \frac{k}{\omega}\mathcal{E}_0$$

$$\frac{1}{\mu}(-ik)B_0 = \sigma\mathcal{E}_0 + (-i\omega)\,\varepsilon\mathcal{E}_0$$

$$\frac{1}{\mu\omega}k^2 = i\sigma + \varepsilon\omega \quad \Rightarrow \quad k^2 = i\omega\mu\sigma + \varepsilon\mu\omega^2$$

Define k as a complex number $k = \beta + i\gamma$

That means: $\vec{E} = \mathcal{E}_0 \vec{e}_x\, e^{i(\beta+i\gamma)z - i\omega t} = \mathcal{E}_0 \vec{e}_x\, e^{-\gamma z}\, e^{i\beta z - i\omega t}$

$\beta = \mathrm{Re}[k]$ — "normal" wave vector

$\gamma = \mathrm{Im}[k]$ → linear absorption

$$k^2 = (\beta+i\gamma)^2 = \beta^2 - \gamma^2 + 2i\beta\gamma = i\omega\mu\sigma + \varepsilon\mu\omega^2$$

$$\begin{cases} \beta^2 - \gamma^2 = \varepsilon\mu\omega^2 \\ 2\beta\gamma = \omega\mu\sigma \end{cases} \Rightarrow \begin{cases} \beta^2 = \dfrac{\omega^2\mu\varepsilon}{2}\left(1 + \sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}\right) \\ \gamma = \omega\mu\sigma / 2\beta \end{cases}$$

Functions of the complex variables.

$$f(x+iy) = u(x,y) + iv(x,y)$$

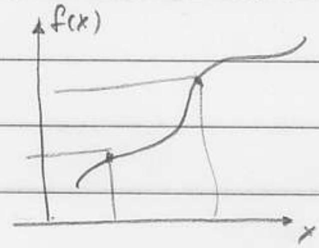
We talked about simple functions last time

$$e^z = e^x(\cos y + i\sin y) \Rightarrow u(x,y) = e^x\cos y, \quad v(x,y) = e^x \sin y$$

$$\operatorname{Ln} z = \ln re^{i\varphi} = \ln r + i\varphi + i2\pi n \qquad n = 0, \pm1, \pm2$$

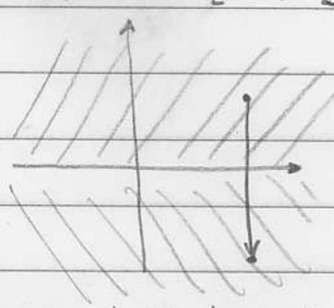
$$u(x,y) = \ln\sqrt{x^2+y^2} \qquad v(x,y) = \tan^{-1} y/x + 2\pi n$$

For real numbers a function makes correspondence b/w two points:



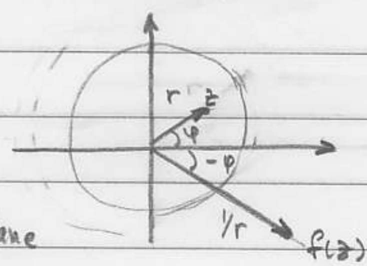
In complex world we need 4D to make a "graph"
So it may be more realistic to talk about "mapping"

a) $z \to z^*$ $\quad u(x,y) = x, \; v(x,y) = -y$

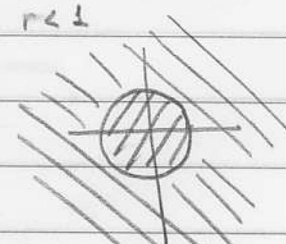


upper half-plane is mapped into a bottom half-plane

b) $f(z) = \frac{1}{z}$ $\qquad z = re^{i\varphi} \Rightarrow \frac{1}{z} = \frac{1}{r}e^{-i\varphi}$

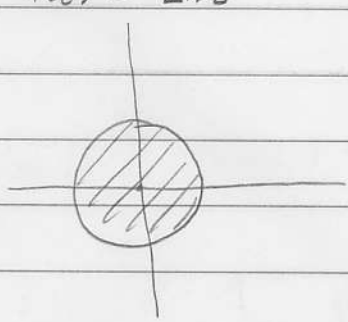


$r < 1$

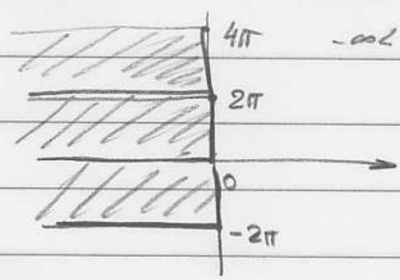


inside of the circle $|r| \leq 1$ is mapped to the outside of the circle

c) $f(z) = \operatorname{Ln} z$



$|r| \leq 1 \qquad 0 < \varphi < 2\pi$



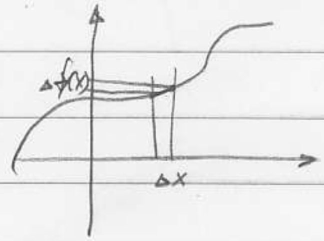
$-\infty < \ln r \leq 0$

inside of the circle $|r| \leq 1$ is mapped to the infinite number of stripes of "width" "2π"

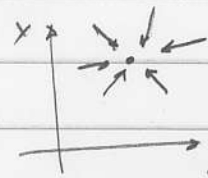
Derivative of complex functions

In the "real" world

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$



In the "complex" world



we can approach any point from many directions.

Analytic functions (= regular = holomorphic = monogenic)

An analytic function has a uniquely defined derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \quad \text{for any } \Delta z$$

Function is analytic in a region if its derivative exists in every point of that region

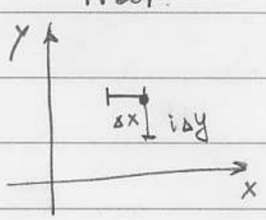
Cauchy - Riemann condition

If $z = x + iy$ and $f(z) = u(x,y) + i v(x,y)$ is analytic

then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ and vice versa.

Proof:

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \qquad \Delta f = f(z+\Delta z) - f(z)$$



*) $\Delta z = \Delta x$

$$\Delta f = u(x+\Delta x, y) + i v(x+\Delta x, y) - u(x,y) - i v(x,y) =$$

$$= \frac{\partial u}{\partial x} \Delta x + i \frac{\partial v}{\partial x} \Delta x \quad \Rightarrow \quad f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

*) $\Delta z = i\Delta y$

$$\Delta f = u(x, y+\Delta y) + i v(x, y+\Delta y) - u(x,y) - i v(x,y) =$$

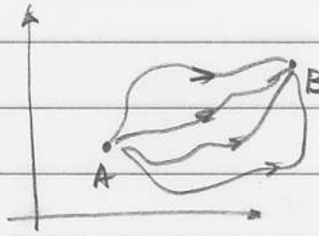
$$= \frac{\partial u}{\partial y} \Delta y + i \frac{\partial v}{\partial y} \Delta y \Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\underset{\overset{\shortparallel}{i\Delta y}}{\Delta z}} =$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Comparing two expressions for the derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;; \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{QED.}$$

Analiticity is important if one takes integrals!

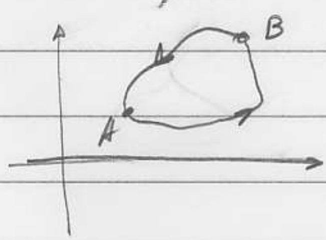


$$\int_A^B f(z)\,dz = \int_A^B (u+iv)(dx+idy) =$$

$$= \int_A^B (u\,dx - v\,dy) + i\int_A^B (u\,dy + v\,dx)$$

If f(z) is analytic

Cauchy theorem



$$\oint f(z)\,dz = 0 \;\Rightarrow\; \int_{\substack{A\\ \text{path 1}}}^{B} f(z)\,dz = \int_{\text{path 2}}$$

$$\oint (u\,dx - v\,dy) = \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

if f(z) is analytic

$$\oint (u\,dy + v\,dx) = \iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$