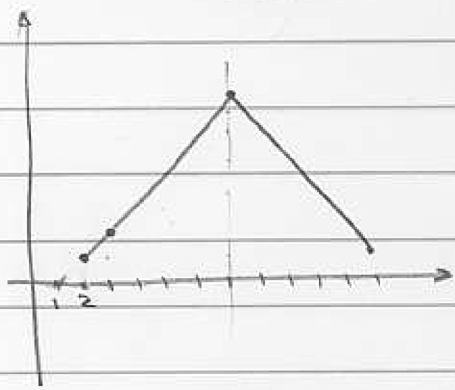
Lecture #24
2009

Random variables

To describe the sample space we have to list all the possible outcomes $\{x_i\}$ and their probabilities $\{p_i\}$. If the number of outcomes is large, it is convenient to describe the associated probability as a function of the outcome $p_i = p(x_i)$

<u>For example</u> sum of two dice. Possible outcomes $\{2 .. 12\}$



$x=2 \quad p_2 = \frac{1}{6}\cdot\frac{1}{6}$

$x=3 \quad p_3 = \frac{1}{36}+\frac{1}{36} = \frac{2}{36}$

...

$x=7 \quad p_7 = (7-1)\cdot\frac{1}{36} = \frac{6}{36}$

...

$x=12 \quad p_{12} = \frac{1}{6}$

$$p(x_i) = \begin{cases} \frac{x_i-1}{36} & 2 \le x_i \le 7 \\ \frac{12-x_i+1}{36} & 7 < x_i \le 12 \\ 0 & \text{othewise} \end{cases}$$

Important characterisics.

<u>Average value</u> $\mu = \bar{x} = \sum_{i=1}^{N} x_i p_i = \sum_{i=1}^{N} x_i \cdot p(x_i)$

Why we describe the average this way?

We can characterize the spread of outromes $\{x_i\}$ from some value x by calculating $\sum_i (x-x_i)^2$

to minimize this value $\frac{\partial}{\partial x} \sum_i (x-x_i)^2 p_i = 2\sum_i (x-x_i)\cdot p_i = 0$

$$x_{opt} \underbrace{\left(\sum p_i\right)}_{=1} - \sum_i x_i p_i = 0$$

$$x_{opt} = \mu = \sum_i x_i p_i$$

The characterization of the spread is called <u>variance</u>

$$Var(x) = \sum_i (x_i - \mu)^2 p_i = \sum_i (x_i-\mu)^2 \cdot p(x_i)$$

Standard deviation $\delta_x = \sqrt{Var(x)}$

Continuous random variable

To go from descrete to a continuous variables we can think of the probability Δp_0 of the value x to fall within x_0 and $x_0+\Delta x$ and then $\Delta x \to 0$

$$\{x_i\} \to x \qquad \{p_i\} \to dp(x) \approx f(x)\,dx$$

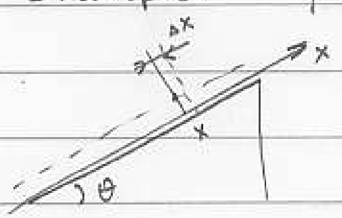
The function $f(x) = \frac{dp}{dx}$ is called probability density

$$p(x_1, x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

Average $\mu = \int_{-\infty}^{+\infty} x\,dp(x) = \int_{-\infty}^{+\infty} x\,f(x)\,dx$

Variance $Var(x) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx = \sigma_x^2$

Example: particle on the incline



The probability to find the particle b/w x and $x+\Delta x$ is proportional to the time the particle spends there

$$\Delta p \sim \Delta t = \frac{\Delta x}{v(x)}$$

Energy conservation $E = \frac{1}{2}mv^2 + mgx\sin\theta = mgl\sin\theta$

where l is max displacement

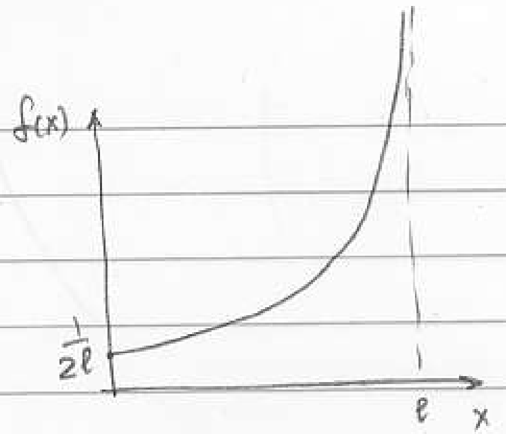
$$v = \sqrt{2g(l-x)\sin\theta}$$

$$\Delta p \sim \frac{\Delta x}{\sqrt{2g(l-x)\sin\theta}} \propto \frac{\Delta x}{\sqrt{l-x}} \qquad f(x) = \frac{\Delta p}{\Delta x} \xrightarrow{x\to 0} \frac{1}{\sqrt{l-x}} \cdot [\text{normalization}]$$

$$f(x) = \frac{N}{\sqrt{l-x}} \qquad \int_0^l f(x)\,dx = N\int_0^l \frac{dx}{\sqrt{l-x}} = 1$$

$$N\,2\sqrt{l-x}\,\Big|_l^0 = 2N\sqrt{l} = 1 \quad N = \frac{1}{2\sqrt{l}}$$

$$f(x) = \frac{1}{2\sqrt{l(l-x)}}$$



average position

$$\mu = \int_0^l \frac{x}{2\sqrt{l(l-x)}}\,dx = \frac{l}{2}\int_0^1 \frac{t\,dt}{\sqrt{1-t}} =$$

$$= \frac{l}{2}\int_0^1 \left(-\sqrt{1-t} + \frac{1}{\sqrt{1-t}}\right)dt = \frac{l}{2}\left(-\frac{2}{3}+2\right) = \frac{2l}{3}$$

Most common probability distributions

<u>Binominal</u> <u>distribution</u> – probability of a certain descrete outcom x in a sequence of n independent trails with an individual success probability p

Examples: a) n coin flips with exactly x heads

$$p_n(x) = \underbrace{\left(\tfrac{1}{2}\right)^x}_{\text{chance of } x \text{ heads}} \underbrace{\left(\tfrac{1}{2}\right)^{n-x}}_{\text{chance of } (n-x) \text{ tails}} \underbrace{C(x,n)}_{\text{possible combinations}} = C(n,x)\left(\tfrac{1}{2}\right)^n$$

b) n dice throws with exactry x ones (or sixes)

$$p_n(x) = \left(\tfrac{1}{6}\right)^x \left(\tfrac{5}{6}\right)^{n-x} C(x,n) = \frac{n!}{x!(n-x)!}\left(\tfrac{1}{6}\right)^n \left(\tfrac{5}{6}\right)^n$$

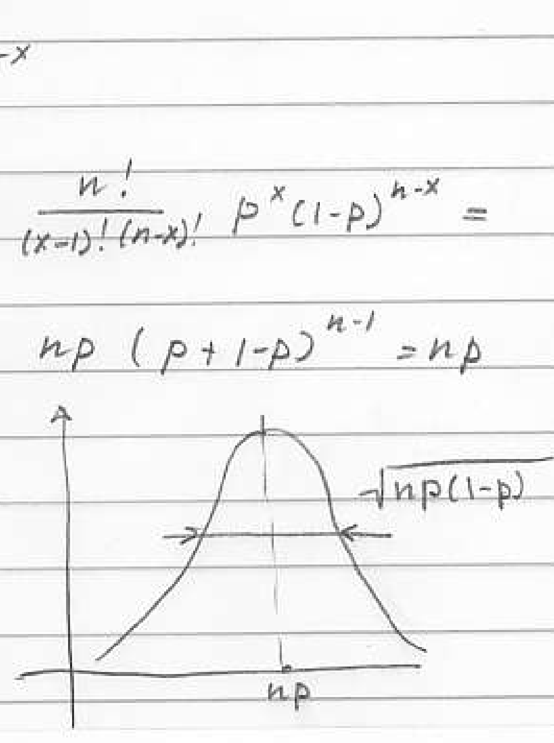
In general, the binomial distribution

$$B_n(x) = \frac{n!}{x!(n-x)!}\, p^x (1-p)^{n-x}$$

Average: $\mu = \langle x \rangle = \sum_{x=0}^{n} x\, B_n(x) = \sum_{x=0}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x} =$

$$= n\cdot p \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1}(1-p)^{n-x} = np\,(p+1-p)^{n-1} = np$$

Similarly $\quad var(x) = np(1-p)$



Poisson distribution — describes events repeated with constant rate of probability (usually the probability is low)
Classic example - radiative decay. Also it describes photon count rate at very low light intensity

$P(t, t+\Delta t) = \mu \cdot \Delta t$ — constant rate

The probability to observe n particle to decay b/w $t=0$ and $t = t+\Delta t$

$$P_n(t+\Delta t) = P_n(t)\,P_0(\Delta t) + P_{n-1}(t)\,P_1(\Delta t) =$$

$$\cancel{P_n(t)} + \frac{\partial P_n}{\partial t}\cdot\Delta t = P_n(t)\,(\cancel{1}-\mu\Delta t) + P_{n-1}(t)\,\mu\Delta t$$

$$\frac{\partial P_n}{\partial t} = -\mu P_n(t) + \mu P_{n-1} = -\mu\,(P_n(t) - P_{n-1}(t))$$

For $n=0$ $\quad \frac{\partial P_0}{\partial t} = -\mu P_0 \quad P_0(t) = e^{-\mu t}$

For $n=1$ $\quad \frac{\partial P_1}{\partial t} = -\mu\,(P_1 - e^{-\mu t}) \quad P_1(t) = \mu t\, e^{-\mu t}$

$$P_n(t) = \frac{(\mu t)^n}{n!}\, e^{-\mu t}$$

The classic Poisson distribution calculates everything "per unit time" $(t=1)$

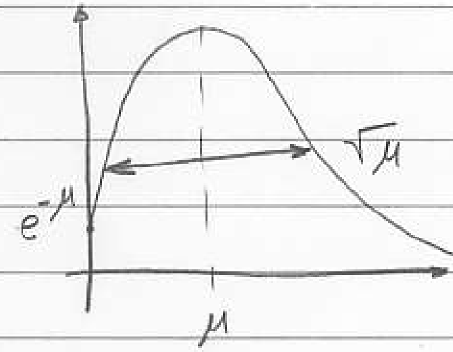
$$P_n = \mu^n/n!\; e^{-\mu}$$

This is the normalized distribution $\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} = e^{\mu} e^{-\mu} = 1$

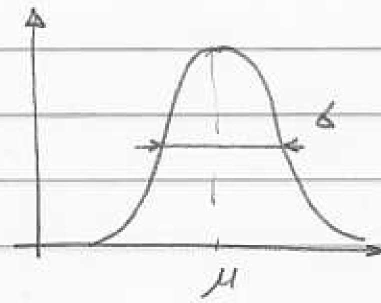
Average number of counts

$$\langle n \rangle = \sum_{n=0}^{\infty} n \frac{\mu^n}{n!} e^{-\mu} = e^{-\mu} \sum_{n=0}^{\infty} \mu \cdot \frac{\mu^{n-1}}{(n-1)!} = \mu$$

$$\sigma^2 = Var(n) = \sum n^2 P_n = \mu$$



Gaussian (Normal) Distribution



$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}$$

μ is the mean value
σ^2 is the variance

Classic example - Maxwell velocity distribution in ideal gas
According to the Boltzman distribution $\frac{N_i}{N} = \frac{e^{-E_i/kT}}{\sum_i e^{-E_i/kT}}$

For atoms in a one-dimentional gas $E_{v_x} = \frac{m v_x^2}{2}$

$$dN(v_x) \propto e^{-m v_x^2/2}\, dv_x$$

$$F(v_x) = \frac{dN(v_x)}{dv_x} \propto e^{-m v_x^2/2}$$

Laws of large numbers

1. For large n and large mean value μ the Poisson distribution approaches Gaussian.
2. In the limit of $n \to \infty$ and $np \to \infty$ the binominal distribution approaches Gaussian.

Proof: $P_n = \frac{\mu^n}{n!} e^{-\mu}$ For large n $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$P_n \approx \frac{\mu^n}{\sqrt{2\pi n}}\, e^{-\mu} \left(\frac{n}{e}\right)^{-n}$$

$$\ln P_n \approx -\mu + n\ln\mu - \ln\sqrt{2\pi n} - n\ln n + n = n - \mu + n\ln\frac{\mu}{n} - \ln\sqrt{2\pi n}$$

The largest contribution is for $n \approx \mu \Rightarrow n = \mu + V$
where μ is large and V/μ is small

$$\ln P_n = \nu + (\mu+\nu)\ln\left(1-\frac{\nu}{\mu+\nu}\right) - \ln\sqrt{2\pi\mu} \approx$$

$$\approx \nu + (\mu+\nu)\left(-\frac{\nu}{\mu+\nu} - \frac{1}{2}\frac{\nu^2}{(\mu+\nu)^2}\right) - \ln\sqrt{2\pi\mu} \approx -\frac{1}{2}\frac{\nu^2}{\mu+\nu} - \ln\sqrt{2\pi\mu}$$

$$P_n = \frac{1}{\sqrt{2\pi\mu}}\, e^{-\frac{(n-\mu)^2}{\mu}}$$

This is the Gaussian distribution with mean value μ and $\sigma^2 = \mu$

The proof for the Binominal distribution is similar

$$B_n(x) = \frac{n!}{x!\,(n-x)!}\, p^x (1-p)^{n-x}$$

Use approximations for large n and large $\mu = np$ and $x = (np+\nu)$ with small ν ($\nu/np \ll 1$)

$$B_n(x) = \frac{1}{\sqrt{2\pi p n (1-p)}}\, e^{-(x-pn)^2/2p(1-p)n}$$

Average pn, variance $np(1-p)$

3. If $n \to \infty$ but $p \to 0$, such that $np \to \mu$ (finite) then Binominal distribution approaches Poisson distr.

$$B_n(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$$

n, $n-x$ are large

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad (n-x)! \approx \sqrt{2\pi n}\left(\frac{n-x}{e}\right)^{n-x}$$

$$\frac{n!}{(n-x)!} = \left(\frac{n}{e}\right)^n \left(\frac{e}{n-x}\right)^{n-x} = \frac{n^{n+x-x}}{\cancel{e^n}} \frac{e^{n-x}}{(n-x)^{n-x}} \approx \left(\frac{n}{e}\right)^x \left(\frac{n}{n-x}\right)^{n-x} \approx$$

$$\approx \left(\frac{n}{e}\right)^x \left(1 + \frac{x}{n-x}\right)^{n-x} \approx \left(\frac{n}{e}\right)^x \cdot e^x \approx n^x$$

average

$$(1-p)^{(n-x)} \sim (1-p)^n \approx \left(1 - \frac{pn}{n}\right)^n \approx e^{-\mu}$$

$$B_n(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \longrightarrow \frac{n^x}{x!} p^x e^{-pn} = \frac{(pn)^x}{x!} e^{-pn} = \frac{\mu^x}{x!} e^{-x}$$

Poisson distribution with $\mu = pn$ (average)