Important information (reminder)

1. <u>Cauchy theorem</u>

If $f(z)$ is analytic inside a closed contour C then $\oint_C f(z)\,dz = 0$

2. <u>Residue theorem</u>

If $f(z)$ is analytic inside C everywhere except a finite number of singularities (poles) then

$$\oint_C f(z)\,dz = 2\pi i \sum_{\text{poles}} \operatorname{Res} f(z)$$

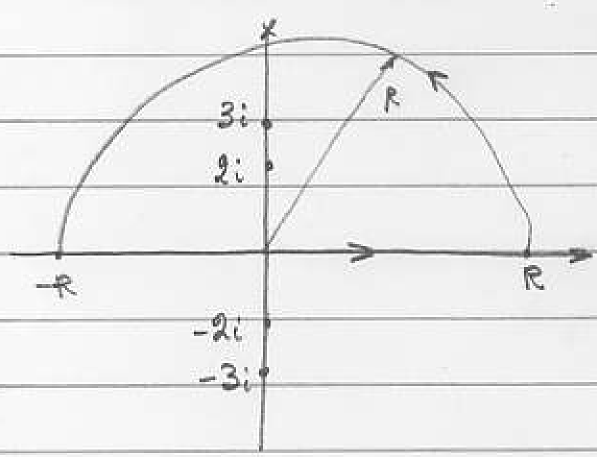
where C is positive (counter clock-wise) contour.

For a simple pole $\operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} f(z)\cdot(z-z_0)$ a residue of $f(z)$ at $z=z_0$

Examples of using Residue theorem for evaluation of integrals.

1. $$\int_{-\infty}^{+\infty} \frac{x^2\,dx}{(x^2+4)(x^2+9)} = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2\,dx}{(x^2+4)(x^2+9)}$$

To make use of the residue theorem, we have to turn this integral into a contour integral. We can do this by adding an extra arc of the radius R to go from R to $-R$



Then our closed circuit C consists of the straight part along x-axis $[-R,R]$ and the arc C_R of radius R.

$$\oint_C \ldots = \int_{-R}^{R} \ldots + \int_{C_R} \ldots$$

The first integral in the limit of $R \to \infty$ is our original integral. Let's check the value of the second integral

$$\int_{C_R} \frac{z^2 dz}{(z^2+4)(z^2+9)} = \left\{ \begin{array}{l} \text{along the arc} \\ z = Re^{i\varphi} \\ dz = iRe^{i\varphi} d\varphi \end{array} \right\} = \int_0^{\pi} \frac{R^2 e^{i2\varphi} \cdot iRe^{i\varphi} d\varphi}{(R^2 e^{i2\varphi} \cancel{+4})(R^2 e^{i2\varphi} \cancel{+9})} \approx$$

$$\approx \frac{1}{R} \int_0^{\pi} ie^{-i\varphi} d\varphi \xrightarrow{R \to \infty} 0$$

Thus the contribution of the second integral is neglegeable.

On the other hand $\oint_C \frac{z^2 dz}{(z^2+4)(z^2+9)} = 2\pi i \operatorname{Res} f(z)$

The function $f(z) = \frac{z^2}{(z^2+4)(z^2+9)}$ has 4 poles $z = \pm 2i$, $z = \pm 3i$, but only 2 of them are inside the circuit

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z \to 2i} \frac{z^2}{(z^2+9)(z+2i)} = \frac{-4}{5 \cdot 4i} = -\frac{1}{5i}$$

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z \to 3i} \frac{z^2}{(z^2+4)(z+3i)} = \frac{-9}{-5 \cdot 6i} = \frac{3}{10i}$$

$$\oint_{\substack{C \\ R \to \infty}} \frac{z^2 dz}{(z^2+4)(z^2+9)} = \int_{-\infty}^{+\infty} \frac{x^2 dx}{(x^2+4)(x^2+9)} = 2\pi i \left(\frac{3}{10i} - \frac{1}{5i} \right) = \frac{\pi}{5}$$

Note: for this integral it does not matter if we close the contour with an arc in the bottom part of the complex plane. In this case the included poles are at $-2i$, $-3i$, and the signs of the residues are reversed. However, in this case the contour direction becomes clock-wise, i.e we will have to add a minus sign to account for that, and the total value of the integral is the same.

2. $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx$

Notice that we have to modify $\cos x$ into e^{ix} → otherwise its integral along the arc C_R will diverge:

$$\cos z = \frac{1}{2}\left(e^{iz}+e^{-iz}\right) = \{z = Re^{i\varphi} = R\cos\varphi + iR\sin\varphi\} =$$

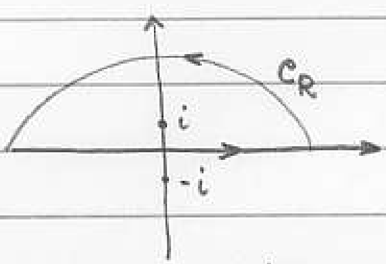
$$= \frac{1}{2}\left(e^{iR\cos\varphi}\cdot e^{-R\sin\varphi} + e^{-iR\cos\varphi}\cdot \underbrace{e^{R\sin\varphi}}_{\text{grows exponentially with } R\to\infty}\right)$$

$$\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \frac{1}{2}\int_0^\infty \frac{e^{ix}+e^{-ix}}{x^2+1}\,dx = \frac{1}{2}\int_0^\infty \frac{e^{ix}}{x^2+1}\,dx + \frac{1}{2}\int_0^{-\infty}\frac{e^{ix}}{x^2+1}\,dx =$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+1}\,dx$$

or

$$\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos x}{x^2+1}\,dx = \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+1}\,dx\right]$$



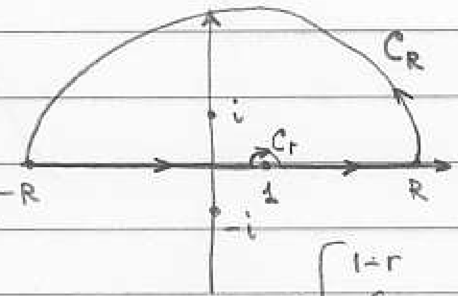
$$\oint_C \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \,\underset{z=i}{\mathrm{Res}}\, \frac{e^{iz}}{z^2+1} = 2\pi i\left(\frac{e^{iz}}{z+i}\right)\Big|_{z=i} =$$

$$= 2\pi i \,\frac{e^{-1}}{2i} = \pi/e$$

$$\oint_C \frac{e^{iz}}{z^2+1}\,dz = \int_{-R}^{R}\ldots + \boxed{\int_{C_R}\frac{e^{iz}}{z^2+1}\,dz} = \int_0^\pi \frac{e^{iR\cos\varphi}\,e^{-R\sin\varphi}}{R^2e^{2i\varphi}\cancel{+1}}\,iRe^{i\varphi}d\varphi \xrightarrow{R\to\infty} 0$$

Thus $\displaystyle\int_{-\infty}^{+\infty}\frac{e^{ix}}{x^2+1}\,dx = \pi/e$ and $\displaystyle\underline{\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \pi/2e}$

3. $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{(x-1)(x^2+1)}\,dx = \mathrm{Im}\left[\int_{-\infty}^{+\infty} \frac{e^{ix}}{(x-1)(x^2+1)}\,dx\right]$



$$\oint_C \frac{e^{iz}}{(z-1)(z^2+1)}\,dz = \int_{C_R}\dots + \int_{-R}^{1-r}\dots + \int_{C_r}\dots + \int_{1+r}^{R}\dots$$

$$\int_{C_R} \frac{e^{iz}}{(z-1)(z^2+1)}\,dz \xrightarrow{R\to\infty} 0$$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\left[\int_{-R}^{1-r}\dots + \int_{1+r}^{R}\dots\right] = \int_{-\infty}^{+\infty}\frac{e^{ix}}{(x-1)(x^2+1)}\,dx$$

$$\lim_{r\to 0}\int_{C_r}\frac{e^{iz}}{(z-1)(z^2+1)}\,dz = \left\{\begin{array}{l} z = 1+re^{i\varphi} \\ dz = ire^{i\varphi}d\varphi \\ e^{iz}\approx e^{i}\end{array}\right\} = \lim_{r\to 0}\int_{\pi}^{0}\frac{e^{i}}{\cancel{re^{i\varphi}}\cdot 2}\, i\cancel{re^{i\varphi}}\,d\varphi = \frac{-i\pi e^{i}}{2}$$

There is only one pole inside C now $z=i$

$$\operatorname*{Res}_{z=i}\frac{e^{iz}}{(z-1)(z^2+1)} = \left.\frac{e^{iz}}{(z-1)(z+i)}\right|_{z=i} = \frac{e^{-1}}{(i-1)\cdot 2i}$$

Thus $\displaystyle\int_{-\infty}^{+\infty}\frac{e^{ix}}{(x-1)(x^2+1)}\,dx - \frac{i\pi e^{i}}{2} = 2\pi\cancel{i}\,\frac{e^{-1}}{\cancel{2i}(i-1)} = \frac{\pi e^{-1}}{i-1}$

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{(x-1)(x^2+1)}\,dx = \frac{i\pi e^{i}}{2} + \frac{\pi e^{-1}}{i-1}$$

$$\int_{-\infty}^{+\infty}\frac{\sin x}{(x-1)(x^2+1)}\,dx = \mathrm{Im}\left[\frac{i\pi e^{i}}{2} + \frac{\pi e^{-1}}{i-1}\right] = \frac{\pi}{2}\cos 1 - \frac{\pi}{2}\,{}^{1}\!/_{e}$$

Notice that $\displaystyle\operatorname*{Res}_{z=1}\frac{e^{iz}}{(z-1)(z^2+1)} = \frac{e^{i}}{2}$, *and* $\displaystyle\frac{i\pi e^{i}}{2} = \frac{1}{2}\left[2\pi i \operatorname*{Res}_{z=1} f(z)\right]$

So some times when the pole is on the contour, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{\text{inside } C}\operatorname{Res} f(z) + \frac{1}{2}\left[2\pi i \sum_{\text{on the boundary}}\operatorname{Res} f(z)\right]$$

4. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx$ function $\dfrac{\ln z}{z^2+1}$ has poles at $z = \pm i$ and $z=0$ is an essential singularity

To complete the circuit we have to extend the integral to $[-\infty, +\infty]$

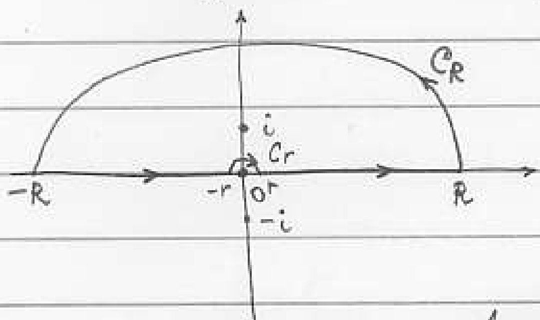
$$\ln(-x) = \ln x + i\pi$$

Thus $\displaystyle\int_{-\infty}^{0} \frac{\ln x}{x^2+1}\,dx \overset{x\to -x}{=} \int_0^{\infty} \frac{(\ln x + i\pi)}{x^2+1}\,dx$

$$\int_{-\infty}^{+\infty} \frac{\ln x}{x^2+1}\,dx = 2\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx + i\pi\int_0^{\infty} \frac{dx}{x^2+1}$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \tan^{-1}(x)\Big|_0^{\infty} = \pi/2$$

$$\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\ln x}{x^2+1}\,dx - \frac{i\pi^2}{4}$$



$$\oint_C \frac{\ln z}{z^2+1}\,dz = \int_{C_R} \ldots + \int_{-R}^{-r} \ldots + \int_{C_r} \ldots + \int_r^R \ldots$$

$$\int_{C_R} \frac{\ln z}{z^2+1}\,dz \sim \frac{\ln R}{R} \xrightarrow{R\to\infty} 0$$

$$\lim_{\substack{R\to\infty \\ r\to 0}} \int_{-R}^{-r} \ldots + \int_r^R \ldots = \int_{-\infty}^{+\infty} \frac{\ln x}{x^2+1}\,dx$$

$$\int_{C_r} \frac{\ln z}{z^2+1}\,dz = \left\{ \begin{array}{l} z = re^{i\varphi} \\ dz = ire^{i\varphi}d\varphi \end{array} \right\} = \int_{\pi}^{0} \frac{\ln r + i\varphi}{1+\cancel{r^2e^{2i\varphi}}} \cdot ire^{i\varphi}\,d\varphi \xrightarrow{r\to 0} 0 \quad \left(\begin{array}{l}\text{since} \\ r\ln r \to 0\end{array}\right)$$

Thus $\displaystyle\oint_C \frac{\ln z}{z^2+1}\,dz = \int_{-\infty}^{+\infty} \frac{\ln x}{x^2+1}\,dx = 2\pi i \cdot \operatorname*{Res}_{z=i} \frac{\ln z}{z^2+i} = 2\pi i \cdot \frac{\ln i}{2i} = \frac{i\pi^2}{4}$

thus $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx = 0$