Problem 1 (15 points)
Using known Taylor series (see equation sheet), calculate

$$\lim_{x\to 0}\left(\frac{e^{-x^2/2}-\cos x}{x^4}\right)$$

$$e^{-x^2/2} = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$

$$\lim_{x\to 0}\left(\frac{e^{-x^2/2}-\cos x}{x^4}\right) = \lim_{x\to 0}\left(\frac{\cancel{1} - \cancel{\frac{x^2}{2}} + \frac{x^4}{8} - \cancel{1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24}}{x^4}\right) = \frac{1}{8} - \frac{1}{24} = \frac{1}{12}$$

Problem 2 (20 points)

Write in (x+iy) form (include all the distinguishable values):

a) $\sqrt[3]{(-1)}$

b) $i^{\ln(i)}$

a) $\sqrt[3]{-1} = \sqrt[3]{e^{i\pi + 2i\pi n}} = e^{i\pi/3 + i\,2\pi n/3}$

Three distinguishable roots

$e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{i\pi/3 + 2\pi i/3} = e^{i\pi} = -1$

$e^{i\pi/3 - i2\pi/3} = e^{-i\pi/3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

b) $\ln i = \underbrace{\cancel{\ln 1}}_{=0} + i\frac{\pi}{2} + i\,2\pi n$

$i = e^{\ln i} \Rightarrow i^{\ln i} = e^{(\ln i)^2} = e^{-(\frac{\pi}{2} + 2\pi n)(\frac{\pi}{2} + 2\pi k)}$

$= e^{-\pi^2/4}\, e^{-\pi^2(n+k+4nk)}$ for $k, n = 0, 1, 2\ldots$

or

$i = e^{i\pi/2}$

$i^{\ln i} = e^{i\pi/2 \cdot (i\frac{\pi}{2} + i2\pi n)} = e^{-\pi^2/4} \cdot e^{-\pi^2 n}$ $\quad n = 0, 1, 2\ldots$

Problem 3 (20 points)
Show that

$$\int_1^\infty (x-1)^{p-\frac{1}{2}} \frac{dx}{x} = \frac{\pi}{\cos \pi p}$$

To use the expression for the B function $x \to \frac{1}{t}$

$$\int_1^0 \left(\frac{1}{t}-1\right)^{p-1/2} \cdot t \left(-\frac{1}{t^2}\right) dt = \int_0^1 (1-t)^{p-1/2}\, t^{-p-1/2}\, dt = B\left(p+\tfrac{1}{2}, -p+\tfrac{1}{2}\right) =$$

$$= \frac{\pi}{\sin \pi(p+\frac{1}{2})} = \frac{\pi}{\cos \pi p}$$

Problem 4 (35 points)

a) Calculate all non-zero associate Legendre functions $P_l^m(\cos\theta)$ for l=0, 1, 2,3.

b) Explain why $P_{2l+1}(0)=0$ for any l.

c) Calculate $P_{12}(0)$ (*Note – deriving $P_{12}(x)$ and then substituting x=0 may not be the fastest way to solve this problem!*)

a) It is convenient to use Rodrigez formular

$$P_{lm}(x) = (1-x^2)^{m/2} \frac{1}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l$$

$$P_0(x) = 1$$

$$P_{10}(x) = x \qquad P_{11}(x) = \sqrt{1-x^2} \qquad P_{1-1}(x) = -\sqrt{1-x^2}$$

$$P_{20}(x) = \frac{1}{8} \frac{d^2}{dx^2} (x^2-1)^2 = \frac{1}{8}(12x^2-4) = \frac{1}{2}(3x^2-1)$$

$$P_{21}(x) = 3x\sqrt{1-x^2} \qquad P_{2-1}(x) = -\frac{1}{2} x\sqrt{1-x^2}$$

$$P_{22}(x) = 3(1-x^2) \qquad P_{2-2}(x) = \frac{1}{8}(1-x^2)$$

$$P_{30}(x) = \frac{1}{48} \frac{d^3}{dx^3} (x^2-1)^3 = \frac{1}{48}(120x^3-72x) = \frac{1}{2}(5x^3-3x)$$

$$P_{31}(x) = \left(\frac{15}{2}x^2 - \frac{3}{2}\right)\sqrt{1-x^2} \qquad P_{3-1}(x) = -\frac{1}{8}(5x^2-1)\sqrt{1-x^2}$$

$$P_{32}(x) = 15x(1-x^2) \qquad P_{3-2}(x) = \frac{1}{8}x(1-x^2)$$

$$P_{33}(x) = 15(1-x^2)^{3/2} \qquad P_{3-3}(x) = -\frac{1}{48}(1-x^2)^{3/2}$$

b) $P_{2l+1}(x) = -P_{2l+1}(-x)$ — odd function

Thus, $P_{2l+1}(0) = 0$

c) $P_{12}(0) = \frac{1}{2^{12}\cdot 12!} \frac{d^{12}}{dx^{12}} (x^2-1)^{12}\Big|_{x=0} = \frac{1}{2^{12}\cdot 12!}\frac{d^{12}}{dx^{12}}\Big(x^{24} - \binom{12}{11}x^{22} + \dots$

$\dots + \underbrace{\binom{12}{6}x^{12}}_{\text{only non-zero term}} + \dots - \binom{12}{1}x^2 + 1\Big)\Big|_{x=0} = \frac{1}{2^{12}\cdot \cancel{12!}}\frac{12!}{6!\cdot 6!}\cdot \cancel{12!} =$

as a result of $\frac{d^{12}}{dx^{12}}x^{12}$

$= \frac{1}{2^{12}}\frac{12!}{6!\,6!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} =$

$= \frac{11\cdot 9\cdot 7\cdot 5\cdot 3}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} = 0.226$

Problem 5 (25 points)

The fraction of light incident on a circular aperture (normal incidence) that is transmitted is given by

$$T = \int_0^{2ka} J_2(x)\left(\frac{2}{x} - \frac{1}{2ka}\right)dx$$

Use the recurrence relations to show that $T = 1 - \frac{1}{2ka}\int_0^{2ka} J_0(x)dx$.

$$\int_0^{2ka} J_2(x)\left(\frac{2}{x} - \frac{1}{2ka}\right)dx = 2\int_0^{2ka} \frac{1}{x} J_2(x)dx - \frac{1}{2ka}\int_0^{2ka} J_2(x)dx =$$

using $\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p} J_{p+1}(x) \Rightarrow \frac{1}{x}J_2(x) = -\frac{d}{dx}\left[\frac{1}{x}J_1(x)\right]$

using $J_{p-1} - J_{p+1} = 2J_p' \Rightarrow J_2 = J_0 - 2J_1'$

$$= -2\int_0^{2ka} \frac{d}{dx}\left[\frac{1}{x}J_1(x)\right]dx - \frac{1}{2ka}\int_0^{2ka}\left(J_0(x) - 2J_1'(x)\right)dx =$$

$$= -2\,\frac{1}{x}J_1(x)\Big|_0^{2ka} + \frac{1}{ka}J_1(x)\Big|_0^{2ka} - \frac{1}{2ka}\int_0^{2ka} J_0(x)dx =$$

$$\lim_{x\to 0}\frac{1}{x}J_1(x) = \lim_{x\to 0}\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)\Gamma(n+2)}\left(\frac{x}{2}\right)^{2n+1} = \left\{\begin{array}{l}\text{only } n=0 \\ \text{is not going} \\ \text{to zero at } x=0\end{array}\right\} = \frac{1}{\Gamma(1)\Gamma(2)}\cdot\frac{1}{2} = \frac{1}{2}$$

$$= \cancel{-\frac{1}{ka}J_1(2ka)} + 1 + \cancel{\frac{1}{ka}J_1(2ka)} - \frac{1}{2ka}\int_0^{2ka} J_0(x)dx = 1 - \frac{1}{2ka}\int_0^{2ka} J_0(x)dx$$

Q.E.D.

Problem 6 (25 points)

The general solutions of Laplace's equation $\nabla^2 u = 0$ in spherical coordinates (r, θ, ϕ) are: $\left\{r^l P_l^m(\cos\theta)(A_{lm}\cos m\phi + B_{lm}\sin m\phi)\right\}$ and $\left\{r^{-l-1} P_l^m(\cos\theta)(A_{lm}\cos m\phi + B_{lm}\sin m\phi)\right\}$.

Find the steady-state temperature distributions inside and outside a sphere of radius a when the surface temperature is $\{3\cos\theta\sin^2\theta\cos 2\phi\}$.

(Hint: your work for the Problem 4a may be handy)

$$\underbrace{3\cos\theta\sin^2\theta}\ \underline{\cos 2\varphi} \qquad m=2$$

$$\frac{1}{5}P_{32}(\cos\theta)\cos 2\varphi \qquad l=3$$

Thus, solution inside

$$u(r,\theta,\varphi) = \left(\frac{r}{a}\right)^3 \frac{1}{5} P_{32}(\cos\theta)\cos 2\varphi$$

and outside

$$u(r,\theta,\varphi) = \left(\frac{a}{r}\right)^4 \cdot \frac{1}{5} P_{32}(\cos\theta)\cos 2\varphi$$

Problem 7 (30 points)
A string of length l is stretched tightly and its ends fastened to supports at $x=0$ and $x=l$. The string has a zero initial displacement and its initial velocity is $v_0(x)= x(l-x)$. Find the displacement $u(x,t)$ by solving wave equation $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$.

A few integrals which you may or may not need to use:

$$\int x^2 \sin kx dx = \frac{1}{k^3}\left(-(kx)^2 \cos kx + 2\cos kx + 2kx \sin kx\right)$$

$$\int x^2 \cos kx dx = \frac{1}{k^3}\left((kx)^2 \sin kx - 2\sin kx + 2kx \cos kx\right)$$

$$\int x \sin kx dx = \frac{1}{k^2}(\sin kx - kx\cos kx)$$

$$\int x \cos kx dx = \frac{1}{k^2}(\cos kx + kx\sin kx)$$

For the one-dimentional string the wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Separating the variables $u(x,t) = X(x)T(t)$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2 \quad \text{(to have oscillatory solutions)}$$

$$\begin{cases} \frac{d^2T}{dt^2} = -v^2k^2T \Rightarrow T(t) = C\cos kvt + D\sin kvt \\ \frac{d^2X}{dx^2} = -k^2X \Rightarrow X = A\cos kx + B\sin kx \end{cases}$$

If ends are fastened $X(0) = X(\ell) = 0$

$A=0, \quad k_n = \frac{\pi n}{\ell}$

$$u(x,t) = \sum_n \left(C_n \cos\frac{\pi n v t}{\ell} + D_n \sin\frac{\pi n v t}{\ell}\right)\sin\frac{\pi n x}{\ell}$$

$u(x,0) = 0 \Rightarrow C_n = 0$

$$u(x,t) = \sum_n D_n \sin\frac{\pi n v t}{\ell}\sin\frac{\pi n x}{\ell}$$

$$\text{Velocity} = \frac{\partial u}{\partial t} = \sum_n D_n \frac{\pi n v}{\ell}\cos\frac{\pi n v t}{\ell}\sin\frac{\pi n x}{\ell}$$

Initial velocity $(t=0)$

$$V_0 = \sum_n \left(D_n \frac{\pi n v}{l} \right) \sin \frac{\pi n x}{l} = x(l-x)$$

To find the expansion coefficient we need to decompose $x(l-x)$ into sine Fourier series

$$x(l-x) = \sum_n b_n \sin \frac{\pi n x}{l}$$

$$b_n = \frac{2}{l} \int_0^l x(l-x) \sin \frac{\pi n x}{l} dx = 2l^2 \int_0^1 t(1-t) \sin \pi n t \, dt =$$

$$= 2l^2 \left[\int_0^1 t \sin \pi n t \, dt - \int_0^1 t^2 \sin \pi n t \, dt \right] = 2l^2 \left[\frac{1}{\pi^2 n^2} (\sin \pi n x - \pi n x \cos \pi n x) \Big|_0^1 - \right.$$

$$- \frac{1}{\pi^3 n^3} \left(-(\pi n x)^2 \cos \pi n x + 2 \cos \pi n x + 2 \pi n x \sin \pi n x \right) \Big|_0^1 = 2l^2 \left[-\cancel{\frac{(-1)^n}{\pi n}} + \cancel{\frac{(-1)^n}{\pi n}} - \right.$$

$$- \frac{2}{\pi^3 n^3} \left[(-1)^n - 1 \right] = \frac{4l^2}{\pi^3 n^3} (1 - (-1)^n) = \frac{8 l^2}{\pi^3 n^3} \quad \underline{n \text{ are odd}}$$

$$D_n \cdot \frac{\pi n v}{l} = \frac{8 l^2}{\pi^3 n^3} \Rightarrow D_n = \frac{8 l^3}{\pi^4 n^4 \cdot v}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8 l^3}{\pi^4 n^4 v} \cos \frac{\pi n v t}{l} \sin \frac{\pi n x}{l}$$

Problem 8 (30 points)
In the initial steady state of a thermo conducting rod of length l the face $x=0$ is at 0^o, and the face $x=l$ is at $T_0=40^o$. From $t=0$ on, both ends are held at $T_0=20^o$. Find the temperature distribution $u(x,t)$ by solving diffusion equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$.

(Some integrals from Problem 7 may be useful.)

Initial steady-state temperature distribution

$$u_{ini} = 40^\circ \, x/l$$

Final steady-state temperature distribution

$$u_{fin} = 20^\circ$$

$$u(x,t) = u_{fin}(x) + u_{trans}(x,t)$$

$$\frac{\partial^2 u_{trans}}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$$

Separate the variables $u_{trans} = X(x)\,T(t)$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha^2}\frac{1}{T}\frac{dT}{dt} = -k^2 \quad \left(\text{to have decaying solution}\right)$$

$$\frac{dT}{dt} = -k^2\alpha^2 T \Rightarrow T = e^{-k^2\alpha^2 t}$$

$$\frac{d^2X}{dx^2} = -k^2 X \Rightarrow X = A\cos kx + B\sin kx$$

Solve for zero boundary conditions

$$X(0) = 0 \Rightarrow A = 0 \qquad X(l) = 0 \quad k = \frac{\pi n}{l}$$

Transient solution

$$u_{trans}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{l}\, e^{-\pi^2 n^2 \alpha^2 t/l^2}$$

and

$$u(x,t) = 20^\circ + \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{l}\, e^{-\pi^2 n^2 \alpha^2 t/l^2}$$

Initial conditions $\quad t=0 \quad u(x,0) = u_{ini}(x)$

$$20^\circ + \sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{\ell} = 40^\circ\, x/\ell$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{\ell} = \underbrace{40^\circ \frac{x}{\ell} - 20^\circ}_{\text{decompose into sine series}}$$

$$B_n = \frac{2}{\ell} \int_0^{\ell} \left(40^\circ \frac{x}{\ell} - 20^\circ\right) \sin \frac{\pi n x}{\ell}\, dx = 80^\circ \int_0^1 t \sin \pi n t\, dt -$$

$$- 40^\circ \int_0^1 \sin \pi n t\, dt = 80^\circ \frac{1}{\pi^2 n^2} \left(\cancel{\sin \pi n t} - \pi n t \cos \pi n t\right)\Big|_0^1 -$$

$$+ 40^\circ \frac{\cos \pi n t}{\pi n}\Big|_0^1 = -\frac{80^\circ}{\pi n}(-1)^n + \frac{40^\circ}{\pi n}(-1)^n - \frac{40^\circ}{\pi n} =$$

$$= -\frac{40^\circ}{\pi n} - \frac{40^\circ}{\pi n}(-1)^n = -\frac{40^\circ}{\pi n}\left[(-1)^n + 1\right] = -\frac{80^\circ}{\pi n}$$

n - even

Solution

$$u(x,t) = 20^\circ + 40^\circ \sum_{n=1}^{\infty} \frac{(-1)^n + 1}{\pi n} \sin \frac{\pi n x}{\ell}\, e^{-\pi^2 n^2 \alpha^2 t/\ell^2} =$$

$$= 20^\circ - 80^\circ \sum_{\substack{n=2 \\ \text{even}}}^{\infty} \frac{1}{\pi n} \sin \frac{\pi n x}{\ell}\, e^{-\pi^2 n^2 \alpha^2 t/\ell^2}$$