

## Non-classical (squeezed state)

Number state:  $|n\rangle$

known number of photons in a mode  $\hat{n}|n\rangle = n|n\rangle$   
 Amplitude is undefined  $\langle n|\hat{E}_x|n\rangle = 0$

$$\Delta E = \sqrt{\langle n|\hat{E}^2|n\rangle - \langle n|\hat{E}|n\rangle^2} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \sqrt{2n+1}$$

Impossible to think about number state as E-M wave, as its phase is completely unknown:  $\Delta n \cdot \Delta \varphi \geq 1$      $\Delta n = 0$      $\Delta \varphi = \infty$

(loose definition, as  $\hat{\varphi}$  cannot be defined)

Number state is an extreme example of a squeezed state, in which we have complete information about one value, while the conjugate is undefined.

Coherent state: closest analogue of a classical e-m wave:  $\hat{a}|d\rangle = d|d\rangle$

$$|d\rangle = e^{-\frac{|d|^2}{2}} \sum_{n=0}^{\infty} \frac{d^n}{\sqrt{n!}} |n\rangle$$

$$\langle d|\hat{E}_x|d\rangle = |d| \cdot 2 \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \sin(\omega t - kz - \theta)$$

$|d|^2$  is average number of photons in  $|d\rangle$   
 Electric field of a single photon

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \text{ for any value of } d$$

Same fluctuations as for the vacuum state  $|n\rangle = 0$

$$\frac{\Delta E_x}{\overline{E_x}} \propto \frac{1}{|d|}$$

The higher is photon number, the less is the effect of fluctuations

$$|d\rangle = \hat{D}(d)|0\rangle$$

$$\hat{D}(d) = e^{\alpha \hat{a}^\dagger - d^* \hat{a}}$$

$$\begin{aligned}
 E_x &= \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} (\hat{a} e^{ikz - i\omega t + \varphi} + \hat{a}^\dagger e^{-ikz + i\omega t + \varphi}) = \\
 &= \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left[ (\hat{a} + \hat{a}^\dagger) \cos(kz - \omega t + \varphi) + i(\hat{a} - \hat{a}^\dagger) \sin(kz - \omega t + \varphi) \right] = \\
 &= 2 \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left[ \underbrace{\frac{1}{2}(\hat{a} + \hat{a}^\dagger)}_{\hat{X}_1} \cos(kz - \omega t + \varphi) - \underbrace{\frac{1}{2i}(\hat{a} - \hat{a}^\dagger)}_{\hat{X}_2} \sin(kz - \omega t + \varphi) \right]
 \end{aligned}$$

With  $\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$  (intensity quadrature)

$\langle \hat{X}_1 \rangle$  or  $\langle X_1 \rangle$   
 $\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$  (phase quadrature)

In general:  $\hat{X}_\alpha = \frac{1}{2}(\hat{a} e^{-i\alpha} + \hat{a}^\dagger e^{i\alpha})$   
 $(\alpha = 0 \rightarrow \hat{X}_1, \alpha = \pi/2 \rightarrow \hat{X}_2)$

We will see in the future that  $\langle X_\alpha \rangle$  and  $\Delta X_\alpha$  can be measured experimentally

$[\hat{X}_1, \hat{X}_2] = \frac{1}{2}i$  (two orthogonal quadratures are not simultaneously measurable)

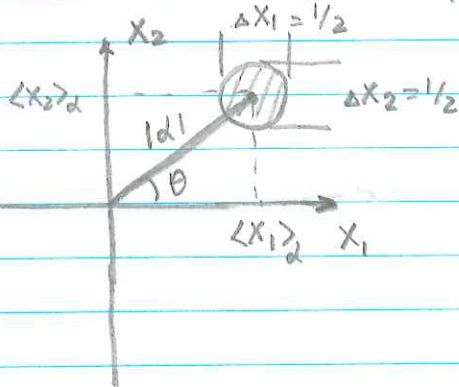
For coherent state

$\langle \hat{X}_1 \rangle_d = \langle d | \hat{X}_1 | d \rangle = \frac{1}{2} \langle d | \hat{a} + \hat{a}^\dagger | d \rangle = \frac{1}{2} (d + d^*) = \text{Re } d$

Phase  $\langle \hat{X}_2 \rangle_d = \text{Im } d$

$\Delta X_1 = \Delta X_2 = \Delta X_\alpha = \frac{1}{2}$   
 ↑  
 for any  $\alpha$

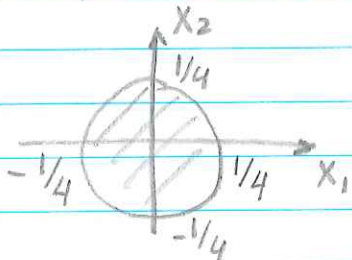
Phase-space picture ("ball on a stick" picture)  
 $d = |d| e^{i\theta}$



The "stick" represents classical (= average) instantaneous value of electric field in polar coordinates

The "ball" represents the gm fluctuations of each quadrature due to their non-communicativity,

(Coherent) vacuum state  $d=0$

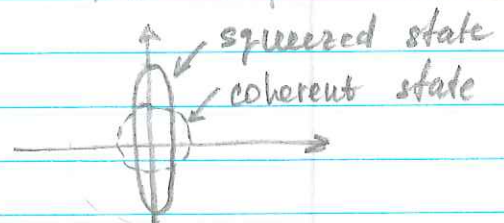


For a coherent state  $\Delta x_1 = \Delta x_2 = \frac{1}{2}$   
 Minimum uncertainty state, that satisfies the quadrature uncertainty relationship

$$\Delta x_1 \cdot \Delta x_2 \geq \frac{1}{4}$$

However, this restricts the product, and one quadrature may have reduced fluctuation at the expense of the other

$$\Delta x_1 < \frac{1}{2} \quad \Delta x_2 > \frac{1}{2}$$



Squeezing operator:

$$\hat{S}(\zeta) = e^{\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})}$$

$$\hat{S}^\dagger(\zeta) = \hat{S}(-\zeta)$$

Baker-Hausdorff lemma:  $\zeta = r e^{i\theta}$

$$\hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) = \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r$$

$$\hat{S}^\dagger(\zeta) \hat{a}^\dagger \hat{S}(\zeta) = \hat{a}^\dagger \cosh r - \hat{a} e^{-i\theta} \sinh r$$

Squeezed vacuum state

$$|\zeta\rangle = \hat{S}(\zeta) |0\rangle$$

$$\langle \zeta | \hat{a} | \zeta \rangle = \langle 0 | \hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) | 0 \rangle = 0$$

$$\langle \zeta | \hat{a}^2 | \zeta \rangle = \langle 0 | \hat{S}^\dagger(\zeta) \hat{a} \hat{a} \hat{S}(\zeta) | 0 \rangle = 0$$

$$= \langle 0 | \hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) \hat{S}^\dagger(\zeta) \hat{a} \hat{S}(\zeta) | 0 \rangle = 0$$

$$\langle \zeta | \hat{X}_{1,2} | \zeta \rangle = 0$$

$$(\Delta X_{1,2})^2 = \frac{1}{4} (\cosh^2 r + \sinh^2 r \mp 2 \sinh r \cosh r \cos \theta)$$

$$= \frac{1}{8} [(e^{2r} + e^{-2r}) \mp (e^{2r} - e^{-2r}) \cos \theta]$$

For  $\theta = 0$

$$(\Delta X_1)^2 = \frac{1}{4} e^{-2r}$$

$$(\Delta X_2)^2 = \frac{1}{4} e^{2r}$$

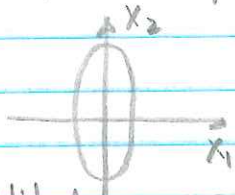
$\theta = \pi$

$$(\Delta X_1)^2 = \frac{1}{4} e^{2r}$$

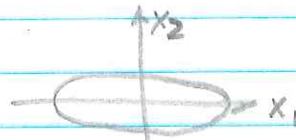
$$(\Delta X_2)^2 = \frac{1}{4} e^{-2r}$$

$$\Delta X_1 \cdot \Delta X_2 = \frac{1}{4}$$

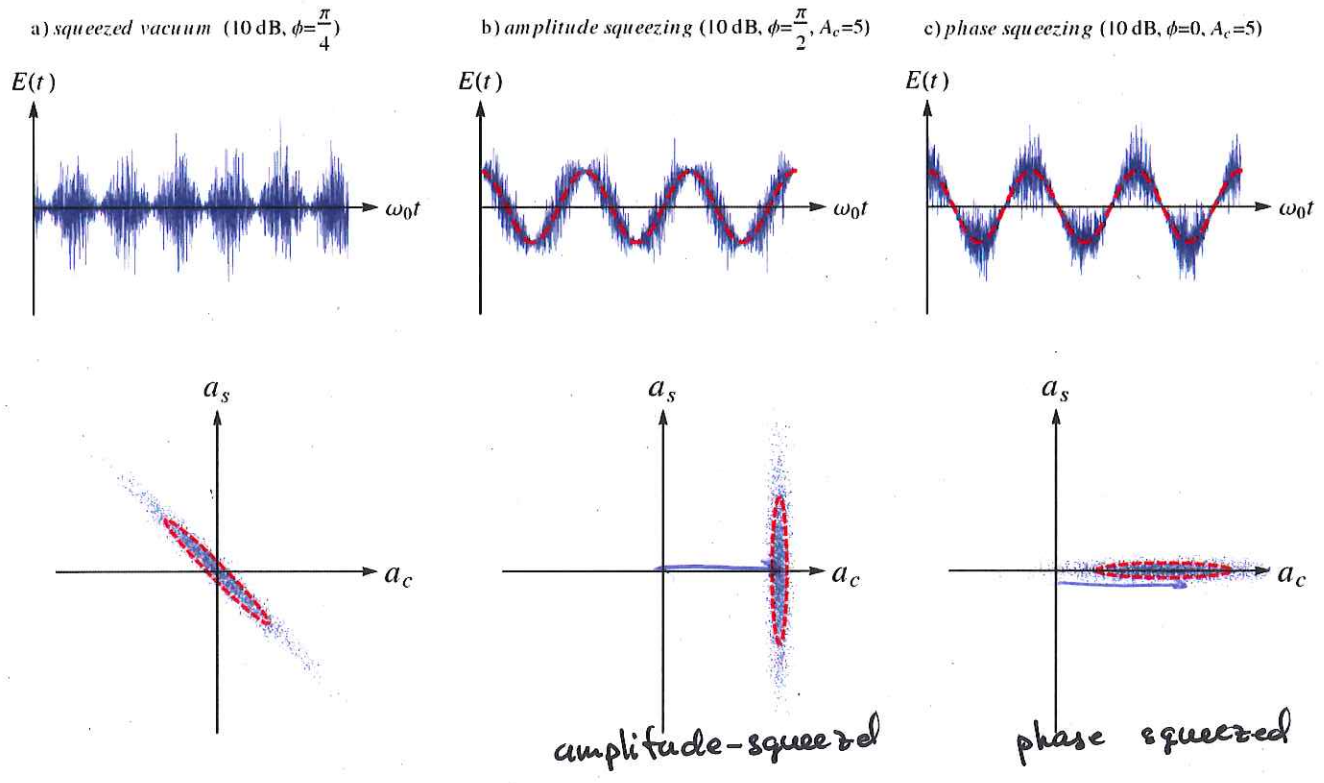
$$\Delta X_1 \cdot \Delta X_2 = \frac{1}{4}$$



amplitude-squeezed



phase-squeezed



### Squeezed vacuum - photon statistics

Energy of a quantized e-m field

$$\hat{H}_{EM} = \hbar\omega(\hat{n} + 1/2)$$

coherent vacuum

$$\langle E \rangle_{\text{vacuum}} = \langle 0 | \hat{H}_{EM} | 0 \rangle = \hbar\omega \langle 0 | \hat{n} + \frac{1}{2} | 0 \rangle = \frac{1}{2} \hbar\omega$$

zero-point energy

for a coherent state

$$\langle E \rangle_{\alpha} = \hbar\omega (|\alpha|^2 + \frac{1}{2})$$

### Squeezed vacuum

$$\begin{aligned} \langle E \rangle_{sv} &= \langle \xi | \hat{H}_{EM} | \xi \rangle = \hbar\omega \langle 0 | \hat{S}^\dagger(\xi) \hat{a}^\dagger \hat{a} | \hat{S}(\xi) | 0 \rangle = \\ &= \hbar\omega \langle 0 | \hat{S}^\dagger(\xi) \hat{a}^\dagger \hat{S}(\xi) \hat{S}^\dagger(\xi) \hat{a} \hat{S}(\xi) | 0 \rangle + \frac{1}{2} \hbar\omega \\ &= \hbar\omega \langle 0 | (\hat{a}^\dagger \cosh r - \hat{a} e^{i\theta} \sinh r) (\hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r) | 0 \rangle = \\ &= \hbar\omega \left[ \langle 0 | (\hat{a}^\dagger)^2 | 0 \rangle (-e^{i\theta} \cosh r \sinh r) + \langle 0 | \hat{a}^2 | 0 \rangle (-e^{-i\theta} \cosh r \sinh r) \right. \\ &\quad \left. + \langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle \cosh^2 r + \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle \sinh^2 r \right] + \frac{1}{2} \hbar\omega \end{aligned}$$

The only non-zero contribution!

$$\langle E \rangle_{sv} = \frac{1}{2} \hbar\omega + \frac{\sinh^2 r}{\text{extra energy due to squeezing}}$$

### Photon number distribution

$$\begin{aligned}
 P_n &= |\langle n | \xi \rangle|^2 = |\langle n | \hat{S}(\xi) | 0 \rangle|^2 = \\
 &= |\langle n | e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})} | 0 \rangle|^2 = \\
 &= \left| \langle n | \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})^k}{k!} | 0 \rangle \right|^2
 \end{aligned}$$

Remarkably,  $P_{2m+1} = 0$   
 only even number of photons  
 can be detected!

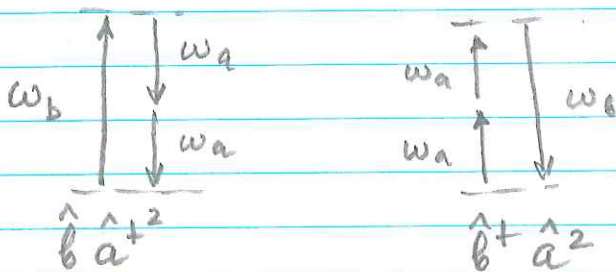
$$P_{2m} = \binom{2m}{m} \frac{1}{\cosh r} \left( \frac{1}{2} \tanh r \right)^{2m} \quad (\text{for } \theta=0)$$

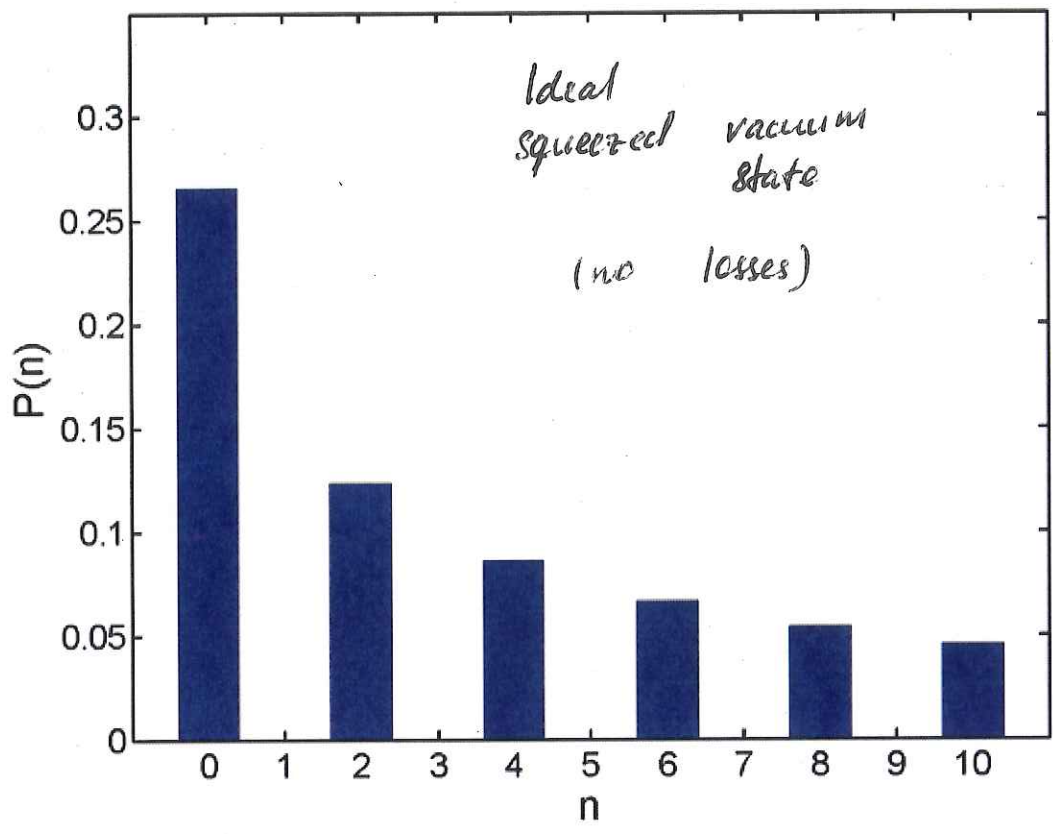
Not too surprising, considering the form  
 of the interaction operator

$$|\psi\rangle_s = e^{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})} |\psi\rangle_0$$

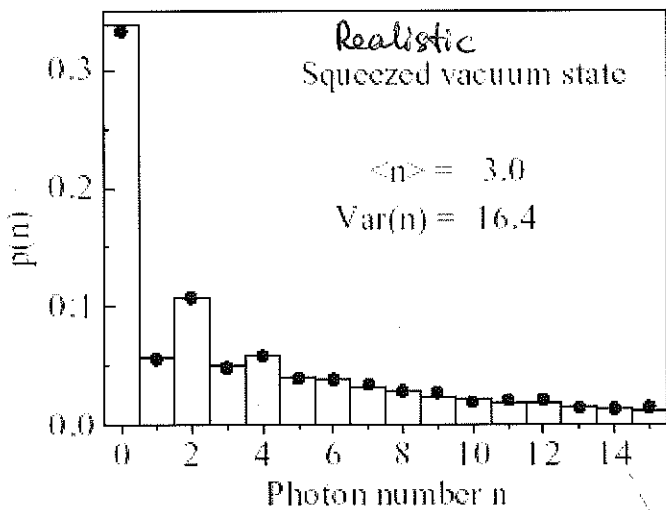
result of the squeezing interaction  
 $\hat{H}_{sq} = \chi (\beta^* \hat{a}^2 - \beta \hat{a}^{\dagger 2})$

Parametric  
 down conversion:  $\hat{H}_{pdc} = \chi_{pdc} (\hat{b}^{\dagger} \hat{a}^2 - \hat{b} \hat{a}^{\dagger 2})$









Any loss of photons  
breaks the symmetry,  
introducing photons into  
odd-number state,  
thus corrupting squeezing