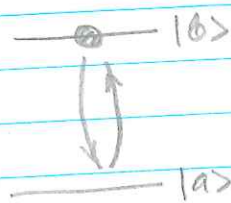
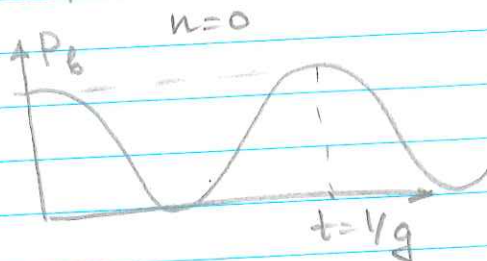


Spontaneous emission Single-mode interaction



$$|\psi(t)\rangle = \cos(g\sqrt{n+1}t)|b\rangle - i\sin(g\sqrt{n+1}t)|a\rangle$$



Vacuum Rabi oscillations

This is not a common scenario, as usually a spontaneous photon is emitted and gone, never to return.

More realistic (common) scenario
atom in a free space, coupled
to infinite number of vacuum
modes

$$\vec{E}_k = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} e^{i\vec{k}\cdot\vec{r}} (\hat{a} + \hat{a}^\dagger)$$

Interaction hamiltonian

$$\hat{H}_{int} = \hbar \sum_k g_k^+ \hat{\delta}_+ \hat{a}_k e^{i(\omega_0 - \omega_k)t} + g_k^- \hat{\delta}_- \hat{a}_k^\dagger e^{-i(\omega_0 - \omega_k)t}$$

here g_k - coupling constant for each mode

Possible quantum state

$$|\psi(t)\rangle = c_0(t)|b, 0\rangle + \sum_k c_{a,k}(t)|a, \pm k\rangle$$

$$c_0(t=0) = 1 \quad c_{a,k}(t=0) = 0$$

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar (c_0 |b, 0\rangle + \sum_k c_k(t) |a, \pm k\rangle) = \hat{H}_{int} |\psi\rangle =$$

$$= \hbar \sum_k g_k (c_0(t) |a, \pm k\rangle e^{-i(\omega_0 - \omega_k)t} + g_k^* c_{a,k} |b, 0\rangle e^{i(\omega_0 - \omega_k)t})$$

$$\begin{cases} \dot{c}_0 = -i \sum_k g_k^* e^{i(\omega_0 - \omega_k)t} c_{a,k}(t) \\ \dot{c}_{a,\pm k} = -i g_k e^{-i(\omega_0 - \omega_k)t} c_0(t) \end{cases}$$

$$c_{a,k}(t) = -i g_k \int_0^t e^{-i(\omega_0 - \omega_k)t'} c_0(t') dt'$$

$$\dot{c}_0 = - \sum_k |g_k|^2 \int_0^t e^{i(\omega_0 - \omega_k)(t-t')} c_0(t') dt'$$

$$\sum_{\mathbf{k}} \rightarrow 2 \frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \left(\int_0^\infty k^2 dk \right) \frac{1}{c^3} \int_0^\infty \omega_k^2 d\omega_k$$

as $k = \frac{\omega}{c}$

$$g_{\pm k} = \frac{1}{\hbar} \sqrt{\frac{\hbar \omega}{2\epsilon_0 V}} \langle a | -\mathbf{d} \cdot \mathbf{e} | b \rangle =$$

$$= \sqrt{\frac{\omega}{2\epsilon_0 \hbar V}} \underbrace{\langle a | -\mathbf{d} | b \rangle}_{P_{ab}} \cos\theta$$

$$\int \cos^2\theta \cdot \sin\theta d\theta = \frac{2}{3}$$

$$\dot{c}_0 = - 2 \frac{V}{(2\pi)^2} \frac{1}{2\epsilon_0 \hbar V} P_{ab}^2 \cdot \frac{2}{3} \cdot \frac{1}{c^3} \times$$

$$\times \int_0^\infty \omega^3 d\omega \int_0^t e^{i(\omega_0 - \omega)(t-t')} c_0(t') dt'$$

$$\int_0^\infty \omega^3 e^{i(\omega_0 - \omega)(t-t')} d\omega \approx \omega_0^3 \int_{0 \rightarrow -\infty}^\infty e^{i(\omega_0 - \omega)(t-t')} d\omega$$

$$\approx \omega_0^3 \cdot 2\pi \delta(t-t')$$

$$\dot{c}_b = -\frac{1}{4\pi\epsilon_0} \frac{4\omega_0^3 p_{ab}^2}{3\hbar c^3} c_b(t) = -\frac{\Gamma}{2} c_b(t)$$

$$\Gamma = \frac{1}{4\pi\epsilon_0} \frac{4\omega_0^3 p_{ab}^2}{3\hbar c^3}$$

the rate of
spontaneous
emission

$$c_b(t) = e^{-\Gamma/2 t}$$

$$P_b(t) = e^{-\Gamma t}$$

the population of the excited state
decays with lifetime $\tau = 1/\Gamma$

Can we say anything about
emitted light?

$$c_{a,k} = -ig_k \int_0^t dt' e^{-i(\omega_0 - \omega_k)t' - \Gamma t'/2}$$

$$= g_k \left[\frac{1 - e^{-i(\omega_0 - \omega_k)t - \Gamma t/2}}{(\omega_k - \omega_0) + i\Gamma/2} \right]$$

$$|\psi(t)\rangle = e^{-\Gamma t/2} |b, 0\rangle + |a\rangle \sum_k g_k \left[\frac{1 - e^{-i(\omega_0 - \omega_k)t - \Gamma t/2}}{(\omega_k - \omega_0) + i\Gamma/2} \right] |1_k\rangle$$

For $t \gg \tau = 1/\Gamma$ we can be fairly
sure the atom is in the ground state

$$|\psi(t \gg \tau)\rangle \approx |a\rangle \sum_k g_k \frac{1}{(\omega_k - \omega_0) + i\Gamma/2} |1_k\rangle$$

one photon is emitted

$$|\delta_{sp}\rangle = \sum_k g_k \frac{1}{(\omega_k - \omega_0) + i\Gamma/2} |1_k\rangle$$

How much light is emitted in each mode?

$$\langle \chi_{sp} | \hat{E}_{k_0}^{(-)} \hat{E}_{k_0}^{(+)} | \chi_{sp} \rangle \propto \langle \chi_{sp} | \hat{n}_{k_0} | \chi_{sp} \rangle = \\ = |g_{k_0}|^2 \frac{1}{(\omega_{k_0} - \omega_0)^2 + \Gamma^2/4}$$

Spectrum of spontaneous emission

