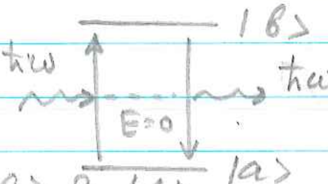
Quantum light – atom interaction in a two-level system

$$\hat H = \underbrace{\sum_i E_i |i\rangle\langle i|}_{\text{atom only}} + \underbrace{\hbar\omega\left(\hat a^+\hat a + \frac{1}{2}\right)}_{\text{photons only}} + \underbrace{\sum_{ij}\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\, p_{ij}|i\rangle\langle j| \times (\hat a + \hat a^+)}_{\text{interaction}}$$

Two-level system



$$\hat H_a = E_a |a\rangle\langle a| + E_b |b\rangle\langle b|$$

let's pick $E=0$ b/w $|a\rangle$ & $|b\rangle$

$$\hat H_a = -\frac{1}{2}(E_b - E_a)|a\rangle\langle a| + \frac{1}{2}(E_b - E_a)|b\rangle\langle b| =$$

$$= \frac{1}{2}\hbar\omega_0 \underbrace{\left(|b\rangle\langle b| - |a\rangle\langle a|\right)}_{\hat\sigma_3 \text{ – inversion operator}} = \frac{1}{2}\hbar\omega_0\hat\sigma_3$$

$$\hat H_i = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\left(p_{ab}|a\rangle\langle b| + p_{ba}|b\rangle\langle a|\right)(\hat a + \hat a^+) =$$

assuming p_{ab} is real

$$= \hbar g\big(\underbrace{|a\rangle\langle b|}_{\hat\sigma_-} + \underbrace{|b\rangle\langle a|}_{\hat\sigma_+}\big)(\hat a + \hat a^+)$$

atomic transition operator

Three atomic operators $\hat\sigma_\pm$ and $\hat\sigma_3$ obey the Pauli matrix commutations

$$[\hat\sigma_+, \hat\sigma_-] = \hat\sigma_3$$

$$[\hat\sigma_3, \hat\sigma_\pm] = 2\hat\sigma_\pm$$

$$\hat H_i = -i\hbar g(\hat\sigma_+ + \hat\sigma_-)(\hat a - \hat a^+)$$

$$\langle\psi|\sigma_+|\psi\rangle = \langle\psi|b\rangle\langle a|\psi\rangle = c_b^* c_a = \rho_{ab}$$

$$\langle\psi|\sigma_3|\psi\rangle = \langle\psi|b\rangle\langle b|\psi\rangle - \langle\psi|a\rangle\langle a|\psi\rangle = |b|^2 - |a|^2 = \rho_{bb} - \rho_{ab}$$

For a two-level system

$$\hat{H} = \frac{1}{2}\hbar\omega_0 \hat{\sigma}_3 + \hbar\omega \hat{a}^+\hat{a} + \hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}^+ + \hat{a})$$

as we discussed last time if $\omega \approx \omega_0$, the two plausible scenarios: photon is aborbed and an atom goes from $|g\rangle \to |e\rangle$, and reversed: photon is emitted and an atom goes from $|e\rangle \to |g\rangle$

$\longrightarrow$ $\cancel{\hat{\sigma}_+\hat{a}^+}$ and $\cancel{\hat{\sigma}_-\hat{a}}$

RWA Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega_0 \hat{\sigma}_3 + \hbar\omega \hat{a}^+\hat{a} + \hbar g(\hat{\sigma}_-\hat{a}^+ + \hat{\sigma}_+\hat{a})$$

no interaction: eigenstates are $\hat{H}|a,n\rangle = -\frac{1}{2}\hbar\omega_0 + \hbar\omega$
$\hat{H}|b,n\rangle = \frac{1}{2}\hbar\omega_0 + \hbar\omega$

We will assume a closed system:

- an atom can only be found in the states $|a\rangle$ & $|b\rangle$ $\quad |a\rangle\langle a| + |b\rangle\langle b| = \mathbb{1}$
- energy is conserved, so the total number of excitations is constant $\hat{N}_e = |b\rangle\langle b| + \hat{a}^+\hat{a}$

Two coupled states

$|b,n\rangle \quad |1\rangle \quad E^{(0)}_{1n} = \frac{1}{2}\hbar\omega_0 + \hbar\omega \cdot n = \hbar\omega(n+\frac{1}{2}) + \frac{1}{2}\hbar(\omega_0 - \omega)$

$|a,n+1\rangle \quad |2\rangle \quad E^{(0)}_{2n} = -\frac{1}{2}\hbar\omega_0 + \hbar\omega(n+1) = \hbar\omega(n+\frac{1}{2}) - \frac{1}{2}\hbar(\omega_0 - \omega)$

$$\Delta = \omega_0 - \omega$$

We can thus decompose the Hamiltonian to the pairs of coupled states for each photon number state

$$\hat{H} = \sum_n \hat{H}_n$$

$$\hat{H}_n = \hbar\omega(n+\tfrac{1}{2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \tfrac{1}{2}\hbar\Delta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} +$$

$$+ \hbar g\begin{pmatrix}0 & \sqrt{n+1}\\ \sqrt{n+1} & 0\end{pmatrix} = \underbrace{\hbar\omega(n+\tfrac{1}{2})\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{\text{overall energy same for both states}} +$$

$$+ \tfrac{1}{2}\hbar\begin{pmatrix}\Delta & 2g\sqrt{n+1}\\ 2g\sqrt{n+1} & -\Delta\end{pmatrix}$$

Eigenstates of the hamiltonian

$$E_{2n} = \hbar\omega(n+1/2) - \frac{1}{2}\hbar\tilde{g}_n$$

$$E_{1n} = \hbar\omega(n+1/2) + \frac{1}{2}\hbar\tilde{g}_n$$

$$\tilde{g}_n = \sqrt{\Delta^2 + 4g^2(n+1)}$$

generalized vacuum Rabi freq

Dressed states

$$|2n\rangle = \cos\theta_n\,|a,n\rangle - \sin\theta_n\,|b,n+1\rangle$$

$$|1n\rangle = \sin\theta_n\,|a,n\rangle + \cos\theta_n\,|b,n+1\rangle$$

$$\cos\theta_n = \frac{\tilde{g}_n - \Delta}{\sqrt{(\tilde{g}_n-\Delta)^2 + 4g^2(n+1)}} \qquad \sin\theta_n = \frac{2g\sqrt{n+1}}{\sqrt{(\tilde{g}_n-\Delta)^2 + 4g^2(n+1)}}$$

For $\Delta = 0$ $(\omega = \omega_0)$

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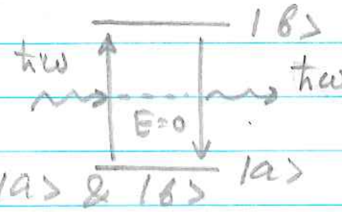
Two-level system



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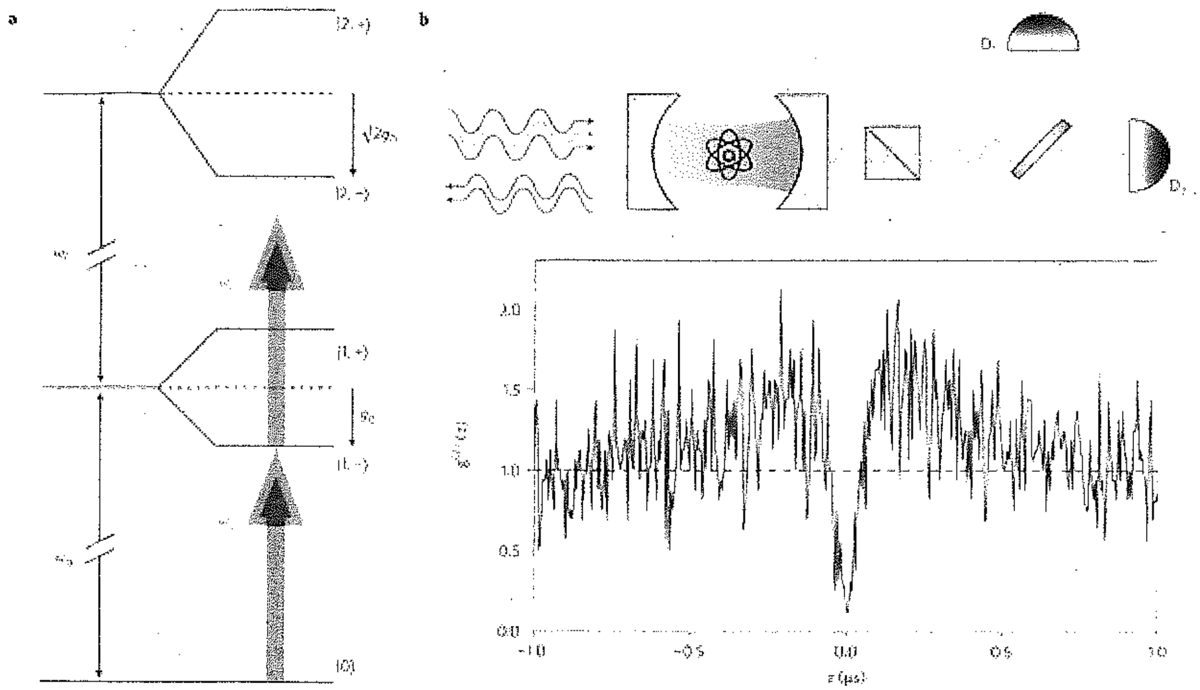
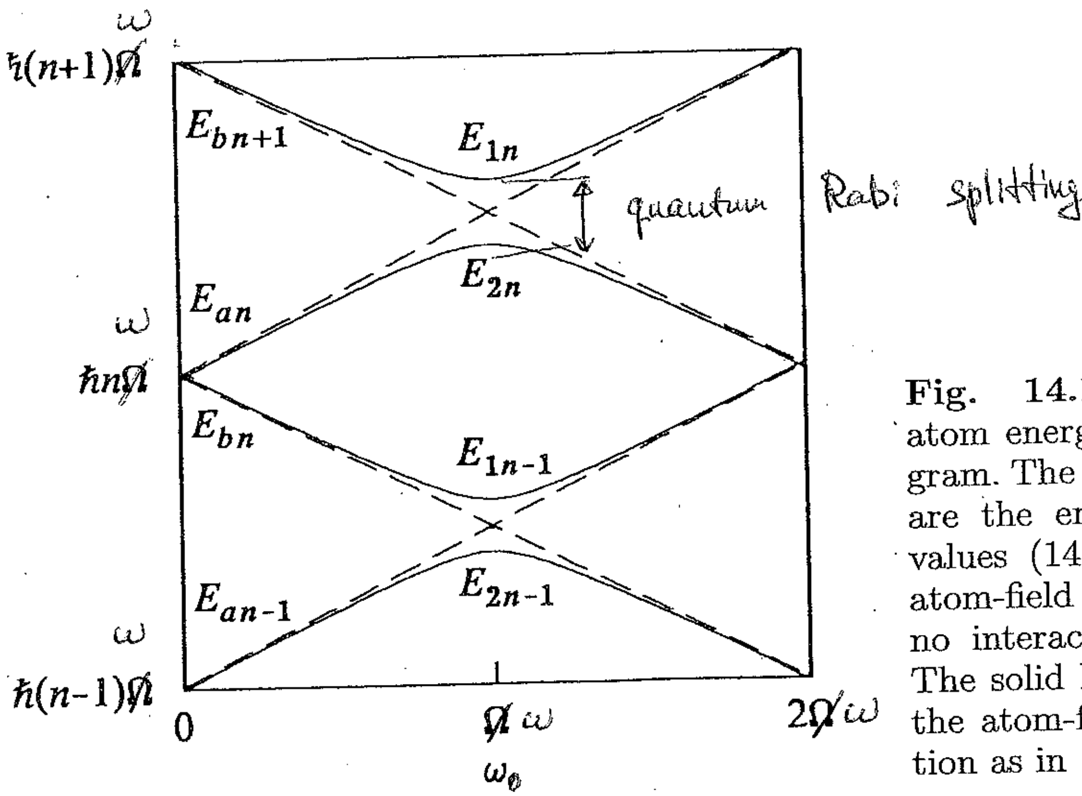




n

Fig. 14.1. Dressed atom energy level diagram. The dashed lines are the energy eigenvalues (14.11) for the atom-field system with no interaction energy. The solid lines include the atom-field interaction as in (14.14)

So, a Fock state with a fixed number of photons $|n\rangle$ behaves very similarly to a classical Rabi flopping.

What about a coherent state?

Initially

$$|\psi_{atom}\rangle_0 = C_a|a\rangle + C_b|a\rangle$$

$$|\psi_{light}\rangle_0 = \sum_{n=0}^{\infty} c_n|n\rangle \qquad c_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$$

for a coherent state

$$|\psi(t=0)\rangle = |\psi_{atom}\rangle_0 |\psi_{light}\rangle_0$$

As we discussed before, light-atom interaction couples states $|a, n+1\rangle$ and $|b, n\rangle$ for all n present.

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}\{[C_b c_n \cos(gt\sqrt{n+1}) - i C_a c_{n+1} \sin(gt\sqrt{n+1})] \times |b\rangle$$
$$+ [-i C_b c_{n-1}\sin(gt\sqrt{n}) + C_a c_n \cos(gt\sqrt{n})]|a\rangle\}|n\rangle$$

For $C_b = 1$ (we start with an atom in the excited state)

$$|\psi(t)\rangle = \sum_{n=0}^{\infty}\{c_n \cos(gt\sqrt{n+1})|b\rangle - i c_{n-1}\sin(gt\sqrt{n})|a\rangle\}|n\rangle$$

$$|\psi_a(t)\rangle = \sum_{n=0}^{\infty} c_n \sin(gt\sqrt{n+1})|n+1\rangle \quad \text{ground}$$

$$|\psi_b(t)\rangle = \sum_{n=0}^{\infty} c_n \cos(gt\sqrt{n+1})|n\rangle \quad \text{excited}$$

Average atomic inversion

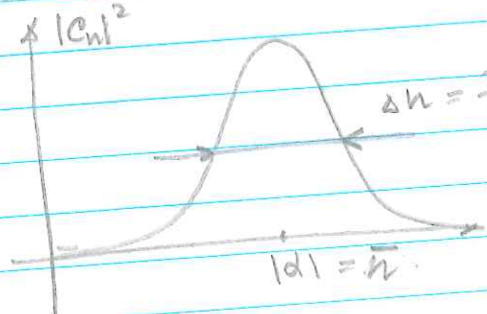
$\langle\psi(t)|\hat{\sigma}_3|\psi(t)\rangle = \langle\psi(t)|\,|b\rangle\langle b| - |a\rangle\langle a|\,|\psi(t)\rangle =$

$= \langle\psi_b|\psi_b\rangle - \langle\psi_a|\psi_a\rangle = \sum_{n=0}^{\infty}|C_n|^2\{(\cos^2 gt\sqrt{n+1}) -$

$- \sin^2(gt\sqrt{n+1})\} = \sum_{n=0}^{\infty}|C_n|^2\cos 2gt\sqrt{n+1} =$

$= e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{n}}{n!}\cos(2gt\sqrt{n+1})$

The output is a combinations of many sine waves with somewhat different periods $\longrightarrow$ no clear Rabi fluctuations



Main contributing components lie between frequencies $2g\sqrt{n-\Delta n}$ and $2g\sqrt{n+\Delta n}$

Corresponding phase spread

$2gt_c(\sqrt{\bar{n}+\Delta n} - \sqrt{\bar{n}-\Delta n}) \approx 2gt_c\sqrt{\bar{n}}\left((1+\frac{\Delta n}{2\bar{n}}) - (1-\frac{\Delta n}{2\bar{n}})\right)$

$\approx 2gt_c\frac{\Delta n}{\sqrt{\bar{n}}} \approx 1 \qquad \Rightarrow gt_c \approx 1$

$t_c \approx 1/g$ depends only on coupling strength

However we can also expect to see a revival of Rabi oscillations if

$$\left(g\sqrt{\bar{n}+1} - g\sqrt{\bar{n}}\right) t_R = 2\pi k \quad (k=0,1,2,\ldots$$

$$g\sqrt{\bar{n}}\left(\cancel{1}+\frac{1}{2\bar{n}}-\cancel{1}\right) t_R = 2\pi \quad \text{(} k=1 \text{ for the first occurance)}$$

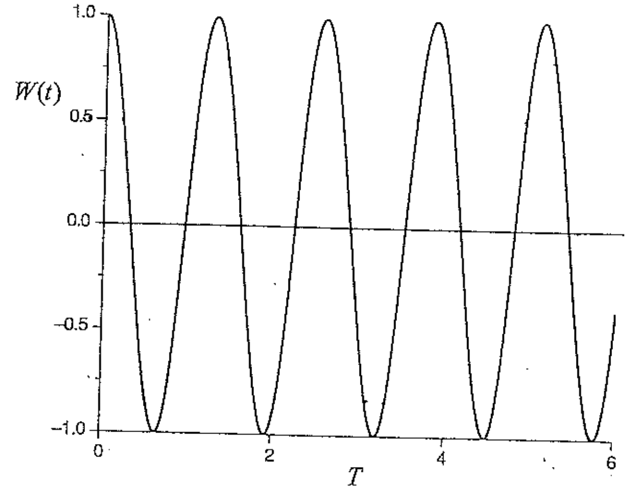
$$g\,\frac{1}{2\sqrt{\bar{n}}}\,t_R = 2\pi$$

$$t_R = \frac{4\pi\sqrt{\bar{n}}}{g}$$

The revival is never "complete" since the frequencies $\{g\sqrt{n}\}$ are not truely equidistant.

Why a coherent state is less "classical" than a number state?
Clear Rabi flopping require knowledge of precise intensity
Coherent state as a minimum uncertainty state, has certain spread in its intensity distribution, that leads to the Rabi flopping diffusion

Fig. 4.6. Periodic atomic inversion with the field initially in a number state $|n\rangle$ with $n = 5$ photons.



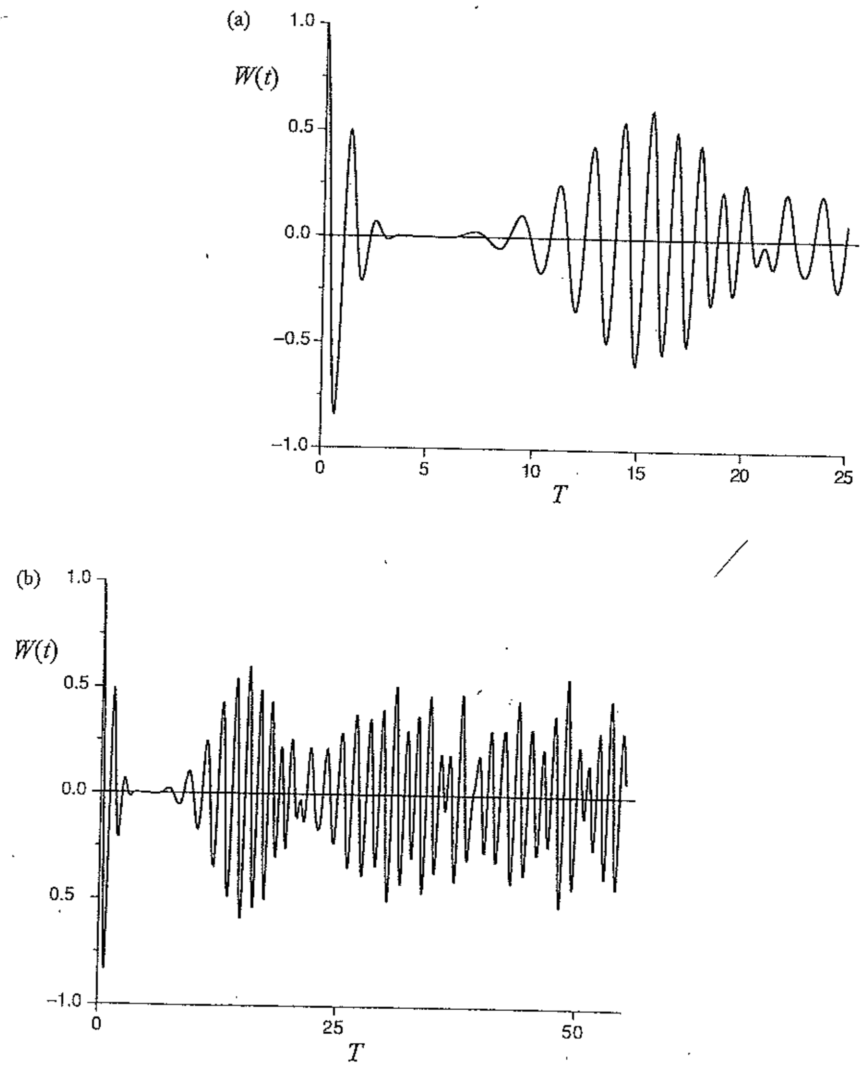


Fig. 4.7. (a) Atomic inversion with the field initially in a coherent state $\bar{n} = 5$. (b) Same as (a) but showing the evolution for a longer time, beyond the first revival. Here, T is the scaled time λt.