Coherent states are the eigenstates of the annihilation operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

If $|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \Rightarrow \hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n \sqrt{n}|n-1\rangle$

$\alpha|\alpha\rangle = \sum_{n=0}^{\infty} \alpha C_n |n\rangle$

$C_{n+1}\sqrt{n+1} = \alpha C_n$

$\Rightarrow C_n = \frac{\alpha^n}{\sqrt{n!}} C_0$

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \langle\alpha|\alpha\rangle = 1 = |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = e^{|\alpha|^2}$$

since $\langle n|n'\rangle = \delta_{nn'}$

$$|C_0|^2 \cdot e^{|\alpha|^2} = 1 \Rightarrow |C_0| = e^{-|\alpha|^2/2} = C_0$$

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle}$$

Average value of the electric field

$$\langle\alpha|E_x|\alpha\rangle = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\underbrace{\langle\alpha|\hat{a}|\alpha\rangle}_{\alpha|\alpha\rangle} e^{ikz-i\omega t} + \underbrace{\langle\alpha|\hat{a}^+|\alpha\rangle}_{\alpha^*\langle\alpha|} e^{-ikz+i\omega t}\right)$$

$$= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\alpha e^{ikz-i\omega t} + \alpha^* e^{-ikz+i\omega t}\right) =$$

$$\alpha = |\alpha| e^{i\varphi} \qquad \alpha^* = |\alpha| e^{-i\varphi}$$

$$= 2|\alpha| \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \cos(kz - \omega t + \varphi)$$

Total energy $\frac{1}{2}\int \varepsilon_0 |E|^2 dV = \underbrace{|\alpha|^2}_{\text{"average" number of photons}} \hbar\omega$

$$\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = |\alpha|^2$$

Electric field fluctuations

$$\langle \alpha | E^2 | \alpha \rangle = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right) \langle \alpha | \hat{a}^2 e^{2(ikz - i\omega t)} + \hat{a}^{+} e^{-2(ikz - i\omega t)} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} | \alpha \rangle$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right) \left(\alpha^2 e^{2(ikz - i\omega t)} + \alpha^{*2} e^{-2(ikz - i\omega t)} + 1 + 2|\alpha|^2 \right) =$$

$$= -\left(\frac{\hbar\omega}{2\varepsilon_0 V}\right) \left(\underbrace{2|\alpha|^2 \cos 2(kz - \omega t + \varphi) + 2|\alpha|^2}_{4|\alpha|^2 \cos^2 (kz - \omega t + \varphi)} + 1 \right)$$

$$\langle \alpha | E^2 | \alpha \rangle = \left(\frac{\hbar\omega}{2\varepsilon_0 V}\right) \left(1 + 4|\alpha|^2 \cos^2 (kz - \omega t + \varphi)\right)$$

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}$$

same fluctuation as in vacuum state

Coherent state is a minimum uncertainty state.

Coherent state is a displaced vacuum state

$$|\alpha\rangle = \hat{D}(\alpha)\, |0\rangle$$

Displacement operator

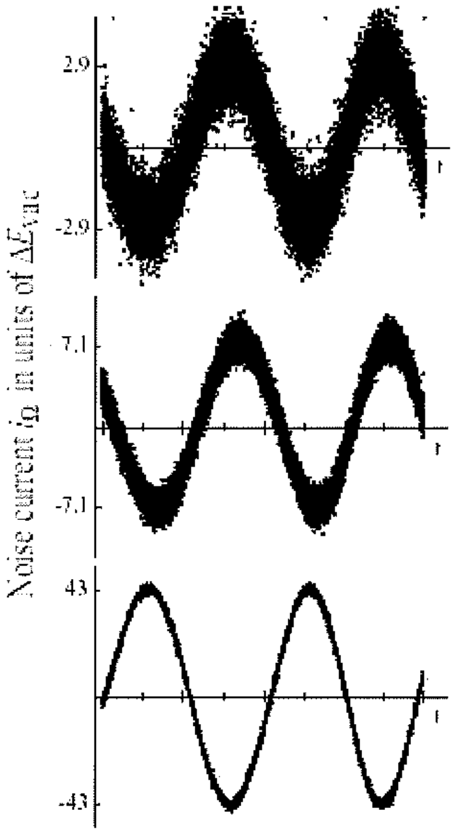
$$\hat{D}(\alpha) = e^{\alpha \hat{a} - \alpha^{*} \hat{a}^{+}}$$

$\hat{D}$ is a unitary operator

$$\hat{D}(\alpha)\, \hat{D}^{+}(\alpha) = \hat{D}^{+}(\alpha)\, \hat{D}(\alpha) = 1$$

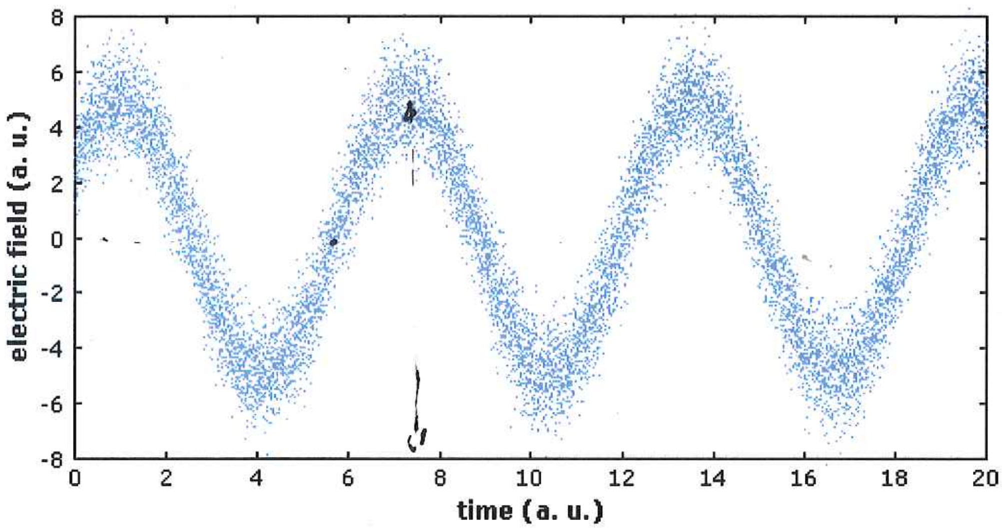
since

$$\hat{D}^{+}(\alpha) = \left(e^{\alpha \hat{a} - \alpha^{*} \hat{a}^{+}}\right)^{+} = e^{\alpha^{*} \hat{a}^{+} - \alpha \hat{a}} = \hat{D}(-\alpha)$$



Increasing strength of e-m field

Since the uncertainty stays the same as amplitude grows, its effect becomes less and less noticable.



Coherent state = "fuzzy" electromagnetic wave

Photon number fluctuations

$$\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}$$

$$\langle \hat{n}^2 \rangle = |\alpha|^4 + |\alpha|^2$$

$$\langle \hat{n} \rangle = |\alpha|^2$$

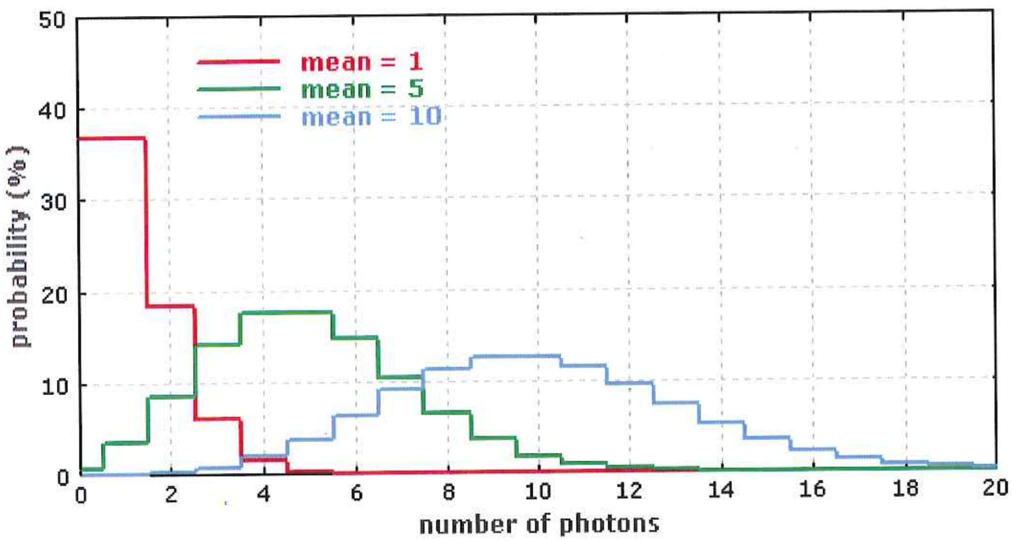
$$\Delta n = |\alpha| = \sqrt{\langle \hat{n} \rangle} = \sqrt{\bar{n}}$$ Poisson process

Photon number distribution

$$p_n = |C_n|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}} \frac{\bar{n}^n}{n!}$$

$$\frac{\Delta n}{n} = \frac{1}{\sqrt{\bar{n}}}$$ shot noise

The stronger is the field ($\bar{n} \gg 1$), the more precisely it is defined, closer to the classical description



Photon distribution in coherent states with different mean value of photons $|\alpha|^2$

Quadrature operators

Electric field

$$E_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left(\hat{a}e^{-i\omega t} + \hat{a}^+ e^{i\omega t}\right)\sin kz$$

Such written $E_x(z,t)$ is a complex operator, so unphysical

$$E_x(z,t) = \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[\hat{a}(\cos\omega t - i\sin\omega t) + \hat{a}^+(\cos\omega t + i\sin\omega t)\right] \times \sin kz =$$

$$= \sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[(\hat{a}+\hat{a}^+)\cos\omega t - i(\hat{a}-\hat{a}^+)\sin\omega t\right] \times \sin kz =$$

$$= 2\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left[\hat{X}_1\cos\omega t + \hat{X}_2\sin\omega t\right] \times \sin kz$$

$$\left.\begin{aligned} \hat{X}_1 &= \frac{1}{2}(\hat{a}+\hat{a}^+) \\ \hat{X}_2 &= \frac{1}{2i}(\hat{a}-\hat{a}^+) \end{aligned}\right] \text{ Quadrature operators}$$

They represent "real" and "imaginary" parts of e-m field, oscillating with frequency ω with $\pi/2$ phase lag between each other.

We can also make connections with canonical position and momentum operators for e-m field

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p})$$

Thus

$$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^+) = \frac{\omega}{\sqrt{2\hbar\omega}}\,\hat{q} \quad \text{(like electric field)}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^+) = \frac{1}{\sqrt{2\hbar\omega}}\,\hat{p} \quad \text{(like magnetic field)}$$

Thus, two quadratures are expected to behave as quantum position and momentum → not commute!

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

That implies $\langle(\Delta\hat{X}_1)^2\rangle\langle(\Delta\hat{X}_2)^2\rangle \geq \frac{1}{16}$

Number state $\langle n|\hat{X}_{1,2}|n\rangle = 0$

$$\langle(\Delta\hat{X}_1)^2\rangle = \langle n|\,\hat{X}_{1,2}^2\,|n\rangle = \frac{1}{4}\langle n|\,\hat{a}^2 + \hat{a}^{+2} + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+|n\rangle$$

$$= \frac{1}{4}\langle n|\,2\hat{a}^+\hat{a}+1\,|n\rangle = \frac{1}{4}(2n+1)$$

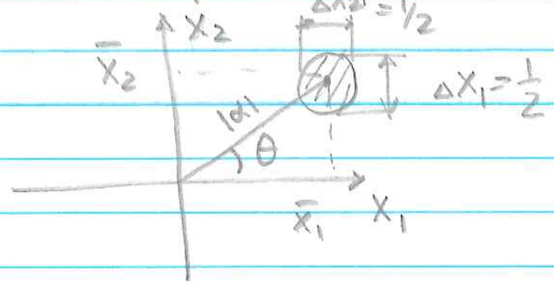
Vacuum state $(n=0)$ $\langle\Delta X_1^2\rangle\langle\Delta X_2^2\rangle = \frac{1}{16}$

min uncertainty state

Coherent state is also a min uncertainty state

$$\langle\Delta X_1^2\rangle = \langle\Delta X_2^2\rangle = \frac{1}{4}$$

Phase-space pictures

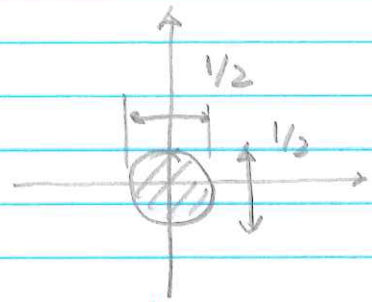


Coherent state

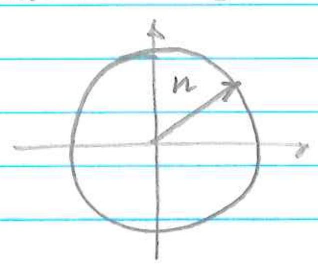
$$\bar{X}_1 = \langle \hat{X}_1 \rangle = \langle \alpha | \tfrac{1}{2}(\hat{a} + \hat{a}^\dagger) | \alpha \rangle = \frac{\alpha + \alpha^*}{2} = \mathrm{Re}(\alpha) = |\alpha| \cos\theta$$

$$\bar{X}_2 = \langle \hat{X}_2 \rangle = \frac{\alpha - \alpha^*}{2i} = \mathrm{Im}(\alpha) = |\alpha| \sin\theta$$

Vacuum state



Number state



Thermal states of e-m field

Thermal radiation is a state of e-m field in thermal equilibrium. Light on its own cannot reach the thermal equilibrium as it does not interact with itself, but when light is brought into contact with material media or generated by thermal source it may thermalize. As a result of such interaction photons of different frequencies are absorbed or emitted, and thus the energy and photon number fluctuates, and thermal field must contain e-m components (modes) of different frequency

Pure vs mixed states

1 particle in a quantum superposition

$$\psi_{1/2} = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

N particles in $\psi_{1/2}$

1/2 1/2 1/2

1/2 1/2

When measured, 50% of particles will be found in state $|1\rangle$, but before the measurement all N-particles are in the same state!

Statistical mixture of particles

1 2 1

2 1 2

Some particles are always in $|1\rangle$, and some always in $|2\rangle$

No wave function is defined

Density matrix operator

$$\hat{\rho} = \sum_n p_n \, |\psi_n\rangle\langle\psi_n|$$

Here $\{|\psi_n\rangle\}$ are the basis states of the system

Pure state $\quad \psi_{1/2} = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \quad \hat{\rho} = |\psi_{1/2}\rangle\langle\psi_{1/2}|$

$$\hat{\rho}_{pure} = \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|) = \frac{1}{2}\left[|1\rangle\langle 1| + |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right]$$

$$\hat{\rho}_{pure} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

quantum coherence
superposition principle!

Mixed state

$p_1 = 1/2 \qquad p_2 = 1/2$

$$\rho_{mixed} = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|2\rangle\langle 2|$$

$$\rho_{mixed} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The probability of finding a system in a state $|\varphi\rangle$ is given by

$$0 \le P_\varphi = \langle\varphi|\hat{\rho}|\varphi\rangle \le 1$$

In oure case the probability of both of our considered systems to be in state $|1\rangle$

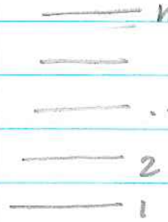
Pure:

$$\langle 1|\hat{\rho}_{pure}|1\rangle = (1\ 0)\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = 1/2$$

Mixed

$$\langle 1|\hat{\rho}_{mixed}|1\rangle = (1\ 0)\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ 0)\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} = 1/2$$

Thermalized harmonic oscillator



Classical case – Boltzman distribution

$$P_n = \frac{e^{-E_n/k_BT}}{\underbrace{\sum_n e^{-E_n/k_BT}}_{Z}} \qquad E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

Quantum case:

$$\hat{\rho}_{th} = \frac{e^{-\hat{H}/k_BT}}{\underbrace{Tr\left[e^{-\hat{H}/k_BT}\right]}_{Z}} \qquad \hat{H} = \hbar\omega\left(\hat{n}+\tfrac{1}{2}\right) = \hbar\omega\left(\hat{a}^+\hat{a}+\tfrac{1}{2}\right)$$

$$Z = Tr\left[e^{-\hat{H}/k_BT}\right] = \sum_{n=0}^{\infty} \langle n| e^{-\hat{H}/k_BT} |n\rangle =$$

$$= \sum_{n=0}^{\infty} e^{-\hat{E}_n/k_BT} = \sum_{n=0}^{\infty} e^{-\hbar\omega(n+1/2)/k_BT} =$$

$$= \frac{e^{-\frac{\hbar\omega}{2k_BT}}}{1-e^{-\hbar\omega/k_BT}}$$

Average number of photons

$$\langle\hat{n}\rangle = Tr\,\langle \hat{n}\hat{\rho}_{th}\rangle = \sum_{n=0}^{\infty} \langle n|\hat{n}\hat{\rho}_{th}|n\rangle =$$

$$= \sum_{n=0}^{\infty} n \underbrace{\langle n|\hat{\rho}_{th}|n\rangle}_{P_n} = \frac{1}{Z}\sum_{n=0}^{\infty} n\, e^{-E_n/k_BT} =$$

$$= \frac{e^{-\frac{\hbar\omega}{2k_BT}}}{Z} \sum_{n=0}^{\infty} n\cdot e^{-\frac{\hbar\omega n}{2k_BT}}$$

$$\sum_{n=0}^{\infty} n e^{-\beta n} = \frac{e^{-\beta}}{(1-e^{-\beta})^2}$$

$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1} \longrightarrow \begin{cases} \hbar\omega \ll k_BT \ \text{(thermal radiation)} \\ \quad \bar{n} \approx k_BT/\hbar\omega \\ \hbar\omega \gg k_BT \ \text{(optical range for room temperature)} \\ \quad \bar{n} \approx e^{-\hbar\omega/k_BT} \end{cases}$$

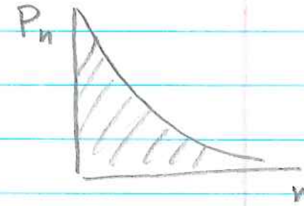
Bose-Einstein statistics

Since $$e^{-\frac{\hbar\omega}{k_B T}} = \frac{\bar{n}}{1+\bar{n}}$$

we can rewrite

$$\hat{\rho}_{th} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}} \right)^n |n\rangle\langle n|$$

$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$$



$$\Delta n = \sqrt{\bar{n} + \bar{n}^2}$$

$$\frac{\Delta n}{n} = \sqrt{1 + \frac{1}{\bar{n}}} \xrightarrow{\bar{n} \gg 1} 1$$

$$\xrightarrow{\bar{n} \ll 1} \infty$$

Total energy density

$$u(\omega) = \hbar\omega \, \bar{n}(\omega) \cdot \underbrace{\rho(\omega)}_{\text{density of states}}$$

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1}$$ Planck's radiation law

One can use it to derive all the laws describing black body radiation.